Topological sensitivity of a shape functional defined from a solution of a high order PDE

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TOPOLOGICAL SENSITIVITY OF A SHAPE FUNCTIONAL DEFINED FROM A SOLUTION OF A HIGH ORDER PDE

AUDRIC DROGOUL[†]

Abstract. The topological gradient is defined as the leading term in the asymptotic expansion of a shape functional with respect to the size of a local perturbation. First introduced by Schumacher [A. Schumacher, *Phd Thesis*, Universitat-Gesamthochschule-Siegen, 1995] and then developped by Sokolowski [J. Sokolowski and A. Zochowski, *SIAM J. Control Optim.*, 37(4), pp. 1251-1272] and Masmoudi [M. Masmoudi, *Computational Methods for Control Applications*, vol. 16, 2001], this notion has been intensively developed in recent years. There are many applications such as in mechanics of structures [S. Amstutz, I. Horchani, and M. Masmoudi, *Control and Cybernetics*, 34(1), pp. 81-101, 2005], in damage evolution modelling [G. Allaire, F. Jouve, and N. Van Goethem, *J. Comput. Phys.*, 230(12), pp. 5010-5044, 2011] and in image processing [L. Jaafar Belaid, M. Jaoua, M. Masmoudi, and L. Siala, *Engineering Analysis with Boundary Elements*, 32(11), pp. 891-899, 2008], [G. Aubert and A. Drogoul, *Control, Optim. Calc. Var.*, to appear]. This paper deals with the topological sensitivity of a cost function involving the m-th derivatives of a function solution of a $2m$ order PDE's with Neumann boundary conditions. We place us in 2D and we consider a domain perturbed by a small crack. Generally the computation of the topological gradient is known up to a polarisation tensor which depends on an exterior problem and on the shape of the perturbation. In this work we reach to fully explicit the topological gradient in function of a direct and an adjoint solution both defined on the unperturbated domain and in function of the normal of the crack. The work is motivated by applications in edge detection (m=1 and m=3) and fine structure detection (m=2) in 2D images.

Key words. Topological gradient, High order PDE, Elliptical equation, 2D imaging, Contours, Fine structures

AMS subject classifications. 35J30, 49Q10, 49Q12, 94A08, 94A13

1. Introduction. The goal of this paper is to generalize the topological gradient method studied and applied in segmentation of images [3, 7, 11, 5] to a more general and higher order problem adapted to object detection. Objects can of Lebesgue measure different to zero. In this case, we are interested in the detection of the boundary of the object commonly called edge. The discontinuity associated to an edge is a discontinuity with a jump of intensity across the structure. Objects can also be of zero Lebesgue measure (filaments, points in 2D) and in this case we want to detect the whole object. Such object is called fine structure and there is no jump of intensity across the structure (see Figure 1).

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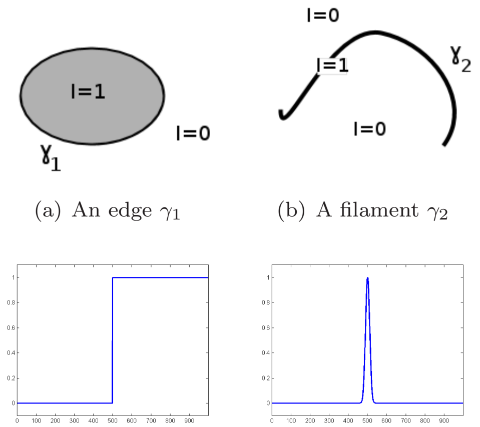


(a) An edge γ_1 (b) A filament γ_2

(c) I across the edge (d) I across the filament

FIG. 1. *Difference between an edge and a fine structure*

In [7, 11], the topological gradient has been applied to a problem of edge detection. It is well known that the detection of edges by using differential operators needs first order derivatives. It is no more true for fine structures which are discontinuities without jump of intensity across the structure. In [5, 9] authors justify theoretically and verify numerically that the detection of fine structures by using differential operators need second order derivatives. Hence they introduce a cost function involving second order derivatives of a regularization of the data solution of a fourth order PDE. In this paper we wonder what does happen for higher order problems with respect to these two kinds of structures ?

In 1D a contour can be modelled by the Heaviside function H. We can approximate H by a regular function $H_{\eta,p} \in C^p(\mathbb{R})$, $p \geq 1$, which writes $H_{\eta,p}(x) = \varphi\left(\frac{x}{\eta}\right) \mathbb{1}_{|x|<\eta} + \mathbb{1}_{x\geq\eta}$ where $\varphi(x) = \frac{1}{2} + \sum_{k=0}^{p} a_k x^{2k+1}$ with $(a_k)_{1\leq k\leq p}$ such that $\varphi(1) = 1$ and $\varphi^{(k)}(1) = 0$ for $k \in [\![1,p]\!]$. Similarly, in 1D a fine structure can be modelled by $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. It can be approximated by $f_\eta(x) = e^{-x^2/\eta^2}$. On Table 1, we study the m-th derivatives of the functions f_1 and $H_{1,4}$ with $m \in [\![0,4]\!]$. We see that derivatives of odd order penalize more edges than fine structures, while derivatives of even order are extremal on fine structures and null on edges. We can generalize this reasoning in 2D by working on transverse cut. Keeping these considerations in mind, in this paper we propose to develop a topological gradient method based on m-th derivatives of a regularized version of the data. From a numerical point of view the high order of the PDE may seem a source of instabilities. However, the generalization of the two cases $[m = 1]$ [3] and $[m = 2]$ [5], is theoretically interesting and shows that the topological gradient can be fully explicit in the case of a straight crack.

Roughly speaking, the topological gradient is performed as follows : let Ω a regular domain of $\mathbb{R}^2$, $j(\Omega) = J(\Omega, u_\Omega)$ be a shape functional with u_Ω solution of a PDE defined on Ω and $J(\Omega, .)$ a cost function depending on Ω. For small $\epsilon > 0$, let $\Omega_\epsilon = \Omega\backslash\overline{\{x_0 + \epsilon\omega\}}$ where $x_0 \in \Omega$ and ω is a given subset of $\mathbb{R}^2$ (typically a crack or a ball). The topological sensitivity of $j(\Omega)$ is given by the leading term in the difference $j(\Omega_\epsilon) - j(\Omega)$ and generally it takes the form : $j(\Omega_\epsilon) - j(\Omega) = \varphi(\epsilon)\mathcal{I}(x_0, \omega) + o\left(\varphi(\epsilon)\right)$ with $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such as $|\varphi| \to 0$ and $\mathcal{I}(x_0, \omega)$ is called the topological gradient associated to the cost function $j(\Omega)$. As said before, in [3, 5] the authors have studied the topological sensitivity of a shape functional of the form $j(\Omega) = J(\Omega, u_\Omega) = \int_\Omega F(u_\Omega, \nabla u_\Omega, \nabla^2 u_\Omega)$ where u_Ω is solution of $Au_\Omega + u_\Omega = f$ where A denotes either the Laplacian or the bilaplacian with Neumann homogeneous boundary conditions and f stands for the data. In this paper we consider a general cost function verifying Hypotheses 1 and defined from the p-th derivatives ($0 \leq p \leq m$) of $2m$-th order's PDE solution. In particular, we apply our general study to quadratic cost functions of the form $J_p(\Omega, u) = \int_\Omega |\nabla^p u|^2$ with $1 \leq p \leq m$. Let us notice that the PDE studied is $(-1)^m \Delta^m u_\Omega + u_\Omega = f$ with Neumann homogeneous boundary conditions and it is associated to the problem $\min_{H^m(\Omega)} J_m(\Omega, u) + \|u - f\|^2_{0,\Omega}$. In image processing, f is generally the observed image possibly degraded by a Gaussian noise and u_Ω can be seen as a regularization of f. A high order problem has already been studied in [4] but with homogeneous Dirichlet conditions of $\partial\Omega$ and the result is given in function of polarization tensors which are not known in general and hard to evaluate. Here, we consider a cracked domain $\Omega_\epsilon = \Omega\backslash\overline{\{x_0 + \epsilon\sigma(\mathbf{n})\}}$, where $\sigma(\mathbf{n})$ is a straight crack centered, the origin and $x_0 \in \Omega$ and ϵ small enough to avoid a possible contact between the perturbation an the boundary of Ω. With these notations we have $\Omega_0 = \Omega$. The paper reads as follows. First we determine the Euler equations associated to the minimization in $H^m(\Omega)$ of the energy function $J_\Omega(u) = \int_\Omega |\nabla^m u|^2$. Then we state the problem and we compute the topological gradient associated to a shape functional verifying Hypotheses 1 an defined from u_{Ω_ϵ} the solution of a $2m$ order PDE defined on Ω_ϵ. The final topological gradient expression is explicit in function of m, u_Ω and an adjoint state v_Ω defined on the unperturbed domain Ω. We give short numerical illustrations in imaging for $m \in \{1, 2, 3\}$ in section 9. For more complete numerical illustrations and applications in imaging we refer the reader to [7, 11] for $m = 1$ and to [9] for $m = 2$. In the sequel, we place us in the local coordinate system to the crack in such a way the center of the perturbation is $x_0 = 0$ and the abscissa axe is given by the crack direction. We consider the function $J_\Omega : H^m(\Omega) \longrightarrow \mathbb{R}$:

$$J_\Omega(u) = \int_\Omega \sum_{k=0}^{m} C_m^k \left(\frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \right)^2 = \int_{\Omega_\epsilon} |\nabla^m u|^2 \tag{1.1}$$

Let $u_\Omega \in H(\Omega)$ defined by

$$u_\Omega = \underset{u \in H(\Omega)}{\operatorname{argmin}} \; J_\Omega(u) + \|u - f\|^2_{0,\Omega} \tag{1.2}$$

where $H(\Omega)$ is the Hilbert space $H(\Omega) = \left\{ u \in L^2(\Omega), \nabla^m u \in L^2(\Omega) \right\}$. Gagliardo-Nirenberg inequalities (see [1] pp 75-79) lead to $H(\Omega) = H^m(\Omega)$.

We define the bilinear form $b_\Omega(u, v) = \frac{1}{2} DJ_\Omega(u).v : H^m(\Omega) \times H^m(\Omega) \longrightarrow \mathbb{R}$ by :

$$b_\Omega(u, v) = \int_\Omega \sum_{k=0}^{m} C_m^k \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}} \tag{1.3}$$

To shorten notations we denote $u_\epsilon = u_{\Omega_\epsilon}$ and define on $H^m(\Omega_\epsilon)$ the bilinear and linear forms $a_\epsilon(u, v)$ and $l_\epsilon(v)$:

$$a_\epsilon(u, v) = b_{\Omega_\epsilon}(u, v) + \int_{\Omega_\epsilon} uv \quad \text{and} \quad l_\epsilon(v) \;\; = \int_{\Omega_\epsilon} fv \tag{1.4}$$

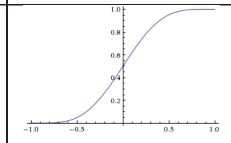
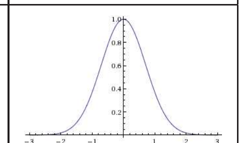
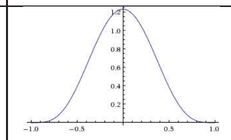
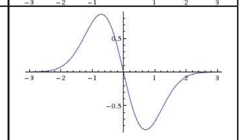
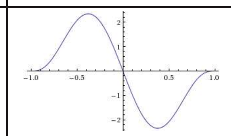
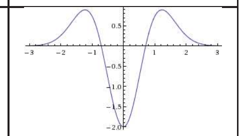
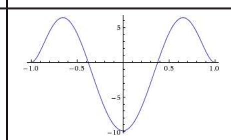
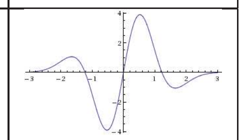
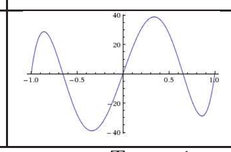
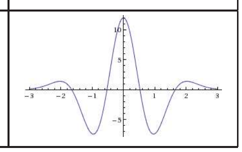
Values of m	Contour	Filament
$m=0$		
$m=1$		
$m=2$		
$m=3$		
$m=4$		

TABLE 1

1D Display of the m-th derivatives of a regularization of a contour and a filament for $m \in [\![0,4]\!]$

Then we introduce the problem $(\mathcal{P}_\epsilon)$ and its solution $u_\epsilon \in H^m(\Omega_\epsilon)$: u_ϵ is the unique solution of

$$a_\epsilon(u_\epsilon, v) = l_\epsilon(v) \qquad \forall v \in H^m(\Omega_\epsilon), \qquad (\mathcal{P}_\epsilon) \tag{1.5}$$

In the sequel we denote by $u_{k,l} = \frac{\partial^{k+l} u}{\partial x_1^k \partial x_2^l}$ for $u \in H^m(\Omega_\epsilon)$ and $k, l \in \mathbb{N}$ such that $k + l \leq m$ and we show that the Euler equations associated to the minimization problem (1.2) in $H^m(\Omega)$ involve the m-th iterates of the Laplace operator Δ^m and m boundary operators given explicitely.

THEOREM 1.1. *Let $b_\Omega(u, v)$ the bilinear form on $H^m(\Omega)$ defined by* (1.3), *we have*

$$b_\Omega(u,v) = (-1)^m \int_\Omega \Delta^m u v + \sum_{i=0}^{m-1} \int_{\partial\Omega} A_i(u) \frac{\partial^{m-1-i} v}{\partial n^{m-1-i}} d\sigma$$

where $A_i : H^m(\Omega) \longrightarrow H^{-i-1/2}(\partial\Omega)$ for $i \in [\![0, m-1]\!]$ is the differential operator of order $i+m$ defined by

$$A_i(u) = \sum_{k=0}^{m} C_m^k \sum_{j=0}^{k-1} (-1)^j \sum_{\substack{0\leq p\leq k-j-1 \\ 0\leq q\leq m-k \\ p+q=i-j}} (-1)^q C_{k-j-1}^p C_{m-k}^q \frac{\partial^{p+q}}{\partial \tau^{p+q}} \left(n_1^{k-j-p+q} n_2^{m-k-q+p} u_{k+j,m-k} \right)$$
$$+ \sum_{0\leq k+l\leq i} C_m^k (-1)^i C_{m-k-l-1}^{i-(k+l)} \frac{\partial^{i-(k+l)}}{\partial \tau^{i-(k+l)}} \left(n_1^{i-(k+l)} n_2^{m-i} u_{2k,m-k+l} \right)$$

with the convention $\sum_{k=0}^{-1} = 0$. *In the case of a straight crack* $\sigma = \{(s,0), -1 < s < 1\}$, *we have :*

(i) For $i \in [\![0, \lfloor \frac{m-1}{2} \rfloor]\!]$

$$A_{2i} = \sum_{k=0}^{i} (-1)^{i-k} C_m^{2k} \left(\sum_{j=0}^{2k-1} C_{2k-j-1}^{k+i-j} C_{m-2k}^{i-k} \right) u_{2(k+i),m-2k} + \sum_{k=0}^{2i} C_m^k u_{2k,m-2k+2i}$$

(ii) For $i \in [\![0, \lfloor \frac{m-2}{2} \rfloor]\!]$ *:*

$$A_{2i+1} = \sum_{k=0}^{i} (-1)^{i-k+1} C_m^{2k+1} \left(\sum_{j=0}^{2k-1} C_{2k-j}^{k+i-j+1} C_{m-2k-1}^{i-k} \right) u_{2(k+i+1),m-(2k+1)} - \sum_{k=0}^{2i+1} C_m^k u_{2k,m-2k+2i+1}$$

Remark 1. *For* $m = 1$, *with these statements* $A_0 = \frac{\partial}{\partial n} = \frac{\partial}{\partial x_2}$.
For $m = 2$, *we have (see the Kirchhoff thin plate equation [8, 5] with Poisson ration* $\nu = 0$*),* $A_0 = B_2$ *and* $A_1 = -B_1$.
In the case of the straight crack $\sigma = \{(s,0), -1 < s < 1\}$, *for* $m \geq 1$, *we have* $A_0 = \frac{\partial^m}{\partial n^m} = \frac{\partial^m}{\partial x_2^m}$, *and* A_i *are homogeneous differential operators of order* $m + i$.

Proof. Sucessive integration by parts and relations

$$\frac{\partial}{\partial x_1} = n_1 \frac{\partial}{\partial n} - n_2 \frac{\partial}{\partial \tau} \quad \text{and} \quad \frac{\partial}{\partial x_2} = n_2 \frac{\partial}{\partial n} + n_1 \frac{\partial}{\partial \tau}$$

give the result. For a complete proof see [10].
□

2. Fundamental solution associated to the m-th Laplacian in 2D. Many solutions of linear differential problems can be expressed by using the fundamental solution associated to the differential operator. In this section we compute the fundamental solution associated to the m-th Laplacian Δ^m.

Theorem 2.1. *Let* $E_m(x)$ *be the fundamental solution of the m-Laplacian defined by*

$$-\Delta^m E_m = \delta_0 \text{ in } D'(\mathbb{R}^2) \tag{2.1}$$

where δ_0 *is the Dirac distribution. Then*

$$E_m(x) = -\frac{1}{2^{2m-1}\pi((m-1)!)^2} |x|^{2(m-1)} log(|x|) \tag{2.2}$$

Proof.
For $m = 1$, $E_1(x) = -\frac{1}{2\pi} log(|x|)$, and for $m = 2$, $E_2(x) = -\frac{1}{8\pi}|x|^2 log(|x|)$.
We search $E_m(x)$ as follows $E_m(x) = a_m |x|^{p_m} log(|x|)$. (2.1) rewrites as $-\Delta^{m-1}(\Delta E_m) = \delta_0$. We deduce that : $\Delta E_m = E_{m-1} + P_{2m-3}(x) = a_{m-1}|x|^{p_{m-1}} log(|x|) + P_{2m-3}(x)$ where $P_{2m-3}(x)$ is a polynomial function of degree less or equal than $2m - 3$. By standard computation : $\Delta\left(|x|^{p_m} log(|x|)\right) = |x|^{p_m - 2} p_m^2 log(|x|) + 2p_m |x|^{p_m - 2}$. Hence

the following relations follow :

$$\begin{cases} p_{m-1} = p_m - 2 \text{ and } p_1 = 0 \\ a_{m-1} = a_m p_m^2 \text{ and } a_1 = -\dfrac{1}{2\pi} \\ p_m - 2 \leq 2m - 3 \end{cases}$$

and we have $p_m = 2(m-1) + p_1$ and $a_m = \frac{a_{m-1}}{p_m^2}$ with $p_1 = 0$. First, we get $p_m = 2(m-1)$ and then $a_m = -\frac{1}{2^{2m-1}\pi((m-1)!)^2}$. We check that $|x|^{p_m-2} = |x|^{2(m-2)}$ is a polynomial function of degree $2m-4 \leq 2m-3$ and we deduce the expression (2.2). $\square$

3. Statements of the problem and notations.

Let $\Sigma \subset \mathbb{R}^2$ a regular open manifold of dimension 1 and $\widetilde{\Sigma}$ a closed and regular curve containing Σ (see Figure 2). We define the following functional spaces :

$$H_{00}^{1/2+i}(\Sigma) = \{u_{|\Sigma}, u \in H^{1/2+i}(\widetilde{\Sigma}), u_{|\widetilde{\Sigma}\backslash\overline{\Sigma}} = 0\}, \quad \forall i \in [\![0, m-1]\!]$$

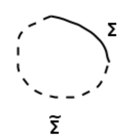


FIG. 2. Σ *and* $\widetilde{\Sigma}$

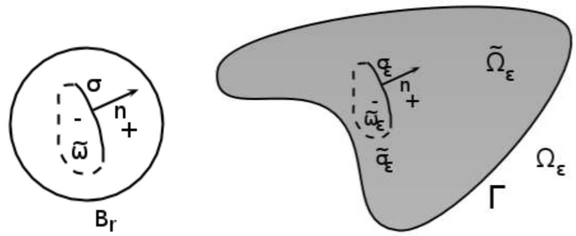
We endow these spaces with the norms :

$$\|u_{|\Sigma}\|_{H_{00}^{1/2+i}(\Sigma)} = \|u\|_{H^{1/2+i}(\widetilde{\Sigma})}$$

Let $\sigma \subset \Omega$ a regular open manifold of dimension 1 containing the origin and of normal $\mathbf{n}$. We denote by $\boldsymbol{\tau}$ the vector such that $(\mathbf{n}, \boldsymbol{\tau})$ be orthonormal. $\partial\tau$ stands for the differentiation in the direction $\boldsymbol{\tau}$ and along σ. in this section we denote by $\Lambda = \mathbb{R}^2\backslash\overline{\sigma}$ the exterior domain of σ and we define the following weighted Sobolev space (see [14]) :

(3.1)
$$W^m(\Lambda) = \left\{ u, \frac{\nabla^k u}{(1+r^2)^{\frac{m-k}{2}} log(2+r^2)} \in L^2(\Lambda), \text{ for } k \in [\![0, m-1]\!], \nabla^m u \in L^2(\Lambda) \right\}$$

where $r = |x|$. $W^m(\Lambda)/\mathbb{P}_{m-1}$ is the quotient space of functions $W^m(\Lambda)$ defined up $\mathbb{P}_{m-1}$ functions. We assume that $\sigma_\epsilon = \{x, \frac{x}{\epsilon} \in \sigma\}$ does not touch $\partial\Omega$; thus we have $\partial\Omega_\epsilon = \sigma_\epsilon \cup \Gamma$. Let $\widetilde{\sigma}$ a closed and regular curve of same dimension of σ such that $\sigma \subset \widetilde{\sigma}$, and let $\widetilde{\omega}$ be the bounded domain of $\mathbb{R}^2$ such that $\partial\widetilde{\omega} = \widetilde{\sigma}$; we denote by $\widetilde{\omega_\epsilon} = \{x, \frac{x}{\epsilon} \in \widetilde{\omega}\}$, $\widetilde{\Omega_\epsilon} = \Omega\backslash\overline{\widetilde{\omega_\epsilon}}$ and we choose $r > 0$ and ϵ small enough such that $\widetilde{\omega} \subsetneq B_r \subset \frac{\Omega}{\epsilon}$ (see Figure 3).



(a) $\widetilde{\sigma} \supset \sigma$ and $B_r \supsetneq \widetilde{\omega}$ (b) Cracked domain

FIG. 3. *Cracked domain and extension of the crack by a closed curve*

For $v \in H^m(\Omega_\epsilon)$ and $u \in H^{2m}(\Omega_\epsilon)$, by using the integration by parts formula given in Theorem 1.1 on $\Omega \backslash \overline{\widetilde{\omega_\epsilon}} \cup \widetilde{\omega_\epsilon}$, we get

$$\int_{\Omega_\epsilon} ((-1)^m \Delta^m u + u)\, v = a_\epsilon(u, v) + \int_{\sigma_\epsilon} \sum_{i=0}^{m-1} A_i(u) \left[\frac{\partial^{m-1-i} v}{\partial n^{m-1-i}}\right] \tag{3.2}$$

where $a_\epsilon(u, v)$ is given in (1.4) and $\left[\frac{\partial^k v}{\partial n^k}\right] = \left(\frac{\partial^k v}{\partial n^k}\right)^+ - \left(\frac{\partial^k v}{\partial n^k}\right)^-$ denotes the jump of $\frac{\partial^k v}{\partial n^k}$ across σ_ϵ, by using notations described in Figure 3. From (1.5) and (3.2), and by assuming that $u_\epsilon \in H^{2m}(\Omega_\epsilon)$, u_ϵ is given by :

$$(\mathcal{P}_\epsilon) \begin{cases} (-1)^m \Delta^m u_\epsilon + u_\epsilon = f, \text{ in } \Omega_\epsilon \\ \qquad\qquad A_i(u_\epsilon) = 0, \text{ on } \sigma_\epsilon \cup \Gamma, \ \forall i \in [\![0, m-1]\!] \end{cases} \tag{3.3}$$

where $f \in L^2(\Omega_\epsilon)$.

We introduce a cost function $J(\Omega, u) : H^m(\Omega) \longrightarrow \mathbb{R}$ verifying

HYPOTHESES 1.

$$J(\Omega_\epsilon, u_\epsilon) - J(\Omega, u_0) = L_\epsilon(u_\epsilon - u_0) + \epsilon^2 \delta J(x_0, \mathbf{n}) + o(\epsilon^2)$$

where $L_\epsilon(u)$ *writes*

$$L_\epsilon(u) = \int_{\Omega_\epsilon} l_0 u + \sum_{1 \le i \le m-1} \int_{\sigma_\epsilon} B_i \left[\frac{\partial^{m-1-i} u}{\partial n^{m-1-i}}\right] + \sum_{0 \le i \le m-1} \int_\Gamma D_i \frac{\partial^{m-1-i} u}{\partial n^{m-1-i}} \tag{3.4}$$

with $\|l_0\|_{0,\Omega_\epsilon} \le C$, $D_i \in H^{-i-1/2}(\Gamma)$, $B_i \in H_{00}^{i+1/2}(\sigma_\epsilon)'$ *and* $\|B_i(\epsilon x)\|_{H_{00}^{i+1/2}(\sigma)'} \le C$ *where* C *is a constant not depending on* ϵ.

To shorten notations, we denote $J_\epsilon(u) = J(\Omega_\epsilon, u)$ and δJ instead of $\delta J(x_0, \mathbf{n})$. In the sequel, to simplify we assume that the crack σ is straight and we assume that $\sigma = \{(s, 0), -1 < s < 1\}$ (we place us in the local coordinate system associated to the crack). We compute the topological gradient by evaluating the leading term with respect to ϵ of the difference $J_\epsilon(u_\epsilon) - J_0(u_0)$ when $\epsilon \to 0$. By using the equations filled by u_ϵ and u_0 and Hypotheses 1, we have :

$$J_\epsilon(u_\epsilon) - J_0(u_0) = L_\epsilon(u_\epsilon - u_0) + \epsilon^2 \delta J + o(\epsilon^2) \tag{3.5}$$

where $L_\epsilon(u)$ is given by (3.4) and we set

$$\mathcal{J}_\epsilon = \epsilon^2 \delta J + o(\epsilon^2) \tag{3.6}$$

To compute (3.5), we introduce $v_\epsilon \in H^m(\Omega_\epsilon)$ solution of the adjoint problem :

$$a_\epsilon(u, v_\epsilon) = -L_\epsilon(u),\ \forall u \in H(\Omega_\epsilon) \tag{3.7}$$

From (3.7), (3.6) and (3.2), then (3.5) writes

$$\begin{aligned}
J_\epsilon(u_\epsilon) - J_0(u_0) &= -a_\epsilon(u_\epsilon - u_0, v_\epsilon) + \mathcal{J}_\epsilon = -l_\epsilon(v_\epsilon) + a_\epsilon(u_0, v_\epsilon) + \mathcal{J}_\epsilon \\
&= -\int_{\Omega_\epsilon} f v_\epsilon + \int_{\Omega_\epsilon} \left((-1)^m \Delta^m u_0 + u_0\right) v_\epsilon - \int_{\sigma_\epsilon} \sum_{i=0}^{m-1} A_i(u_0) \left[\frac{\partial^{m-1-i} v_\epsilon}{\partial n^{m-1-i}}\right] + \mathcal{J}_\epsilon \\
&= -\int_{\sigma_\epsilon} \sum_{i=0}^{m-1} A_i(u_0) \left[\frac{\partial^{m-1-i} v_\epsilon}{\partial n^{m-1-i}}\right] + \mathcal{J}_\epsilon
\end{aligned}$$

By setting $w_\epsilon = v_\epsilon - v_0$ with v_ϵ and v_0 given by (3.7) for $\epsilon > 0$ and $\epsilon = 0$; we rewrite $J_\epsilon(u_\epsilon) - J_0(u_0)$ in function of w_ϵ :

$$J_\epsilon(u_\epsilon) - J_0(u_0) = -\int_{\sigma_\epsilon} \sum_{i=m}^{2m-1} A_i(u_0) \left[\frac{\partial^{2m-1-i} w_\epsilon}{\partial n^{2m-1-i}}\right] + \mathcal{J}_\epsilon \tag{3.8}$$

Then, a change of variables and subscripts give
(3.9)

$$\begin{aligned}
J_\epsilon(u_\epsilon) - J_0(u_0) &= -\sum_{i=0}^{m-1} \epsilon \int_\sigma A_i(u_0)(\epsilon X) \left[\frac{\partial^{m-1-i} w_\epsilon}{\partial n^{m-1-i}}(\epsilon X)\right] d\sigma + \mathcal{J}_\epsilon \\
&= -\sum_{i=0}^{m-1} \epsilon \int_\sigma A_i(u_0)(\epsilon X) \frac{1}{\epsilon^{m-1-i}} \left[\frac{\partial^{m-1-i} \left(w_\epsilon(\epsilon X)\right)}{\partial n^{m-1-i}}\right] d\sigma + \mathcal{J}_\epsilon \\
&= -\sum_{k=0}^{m-1} \epsilon^{1-k} \int_\sigma A_{m-1-k}(u_0)(\epsilon X) \left[\frac{\partial^k}{\partial n^k} \left(w_\epsilon(\epsilon X)\right)\right] d\sigma + \mathcal{J}_\epsilon = -\sum_{k=0}^{m-1} \mathcal{I}_k + \mathcal{J}_\epsilon
\end{aligned}$$

where $\mathcal{I}_k$ for $k \in [\![0, m-1]\!]$ are defined by :

$$\mathcal{I}_k = \epsilon^{1-k} \int_\sigma A_{m-1-k}(u_0)(\epsilon X) \left[\frac{\partial^k}{\partial n^k} \left(w_\epsilon(\epsilon X)\right)\right] d\sigma \tag{3.10}$$

Now, we need to establish the asymptotic expansion of w_ϵ in $H^m(\Omega_\epsilon)$ norm. To do that, first we search for the leading terms in w_ϵ which need to be compensated in order to have an asymptotic expansion in $o(\epsilon)$ in $H^m(\Omega_\epsilon)$ norm (see section 5).

4. Estimations of $A_i(v_0)(\epsilon X)$ for $X \in \sigma$. The following lemma gives the expansion with respect to ϵ at 0 of $A_i(v_0)(\epsilon X)$ for $X \in \sigma$.

LEMMA 4.1. *Let v_0 solution of (3.7) with $\epsilon = 0$. In the case of a straight crack, assuming that v_0 is regular (or equivalently that f is regular), we have the following estimations :*

$$A_0(v_0)(\epsilon X) = g_0(X) + O(\epsilon) \tag{4.1a}$$

$$A_i(v_0)(\epsilon X) = g_i(X) + O(1),\ \forall i \in [\![1, m-1]\!] \tag{4.1b}$$

with

$$g_0(X) = \frac{\partial^m v_0}{\partial x_2^m}(0) \quad \textit{and} \quad g_i(X) = 0,\ \textit{for } 1 \leq i \leq m-1 \tag{4.2}$$

Proof. With the expression of the A_i given in Theorem 1.1 for $i \in [\![1, m-1]\!]$ for a straight crack, and by a Taylor expansion at 0 of $\frac{\partial^{k+l} v_0}{\partial x_1^k \partial x_2^l}$, we get (4.1b). By using the expression of A_0 (see Remark 1) we deduce that :

$$A_0(v_0)(\epsilon X) = \frac{\partial^m v_0}{\partial x_2^m}(\epsilon X)$$

We conclude with a Taylor expansion of $\frac{\partial^m v_0}{\partial x_2^m}(\epsilon X)$ at 0. □

5. Asymptotic expansion of w_ϵ in $H^m(\Omega_\epsilon)$ norm. In this section we do the asymptotic expansion of w_ϵ with respect to ϵ in the sense of the $H^m(\Omega_\epsilon)$ norm. We recall that $w_\epsilon = v_\epsilon - v_0$ is solution of :

$$(\mathcal{Q}_\epsilon^c) \begin{cases} (-1)^m \Delta^m w_\epsilon + w_\epsilon = 0, \text{ in } \Omega_\epsilon \\ A_0(w_\epsilon) = -A_0(v_0), \text{ on } \sigma_\epsilon \\ A_i(w_\epsilon) = -A_i(v_0) - B_i, \text{ on } \sigma_\epsilon,\ \forall i \in [\![1, m-1]\!] \\ A_i(w_\epsilon) = 0, \text{ on } \Gamma,\ \forall i \in [\![0, m-1]\!] \end{cases} \tag{5.1}$$

To estimate w_ϵ we introduce the solution of the exterior problem $R \in W^m(\Lambda)/\mathbb{P}_{m-1}$:

$$(\mathcal{R}_{ext}^c) \begin{cases} \Delta^m R = 0, \text{ in } \Lambda \\ A_i(R) = -g_i, \text{ on } \sigma,\ \forall i \in [\![0, m-1]\!] \end{cases} \tag{5.2}$$

where $\forall i \in [\![0, m-1]\!]$, $g_i \in H_{00}^{1/2+i}(\sigma)'$ is given by (4.2). Thanks to Theorem 10.6 given in Appendix, we deduce that the problem $(\mathcal{R}_{ext}^c)$ has a unique solution $R \in W^m(\Lambda)/\mathbb{P}_{m-1}$ which writes as follows :

$$R(x) = \sum_{i=0}^{m-1} \oint_\sigma \lambda_i(y) A_{i,y}(E(x-y)) d\sigma_y$$

where $\oint$ denotes the Cauchy principal value. Moreover we have :

$$(-1)^{m+1} \left[\frac{\partial^{m-1-i} R}{\partial_n^{m-1-i}} \right] = \lambda_i \quad \forall i \in [\![0, m-1]\!] \tag{5.3}$$

where

$$\begin{cases} \lambda_0(s) = \dfrac{(-1)^{m+1} 2^{2m-1}}{(2m-1) C_{2(m-1)}^{m-1}} \beta \sqrt{1-s^2} \quad \forall (s,0) \in \sigma \\ \lambda_i(s) = 0 \quad \forall i \in [\![1, m-1]\!] \quad \forall (s,0) \in \sigma \end{cases} \tag{5.4}$$

with $\beta = \frac{\partial^m v_0}{\partial x_2^m}(0)$. Thanks to Lemma 10.8 (see Appendix), we get

$$w_\epsilon = \epsilon^2 R\left(\frac{x}{\epsilon}\right) + e_\epsilon \text{ with } \|e_\epsilon\|_{H^m(\Omega_\epsilon)} = O(\phi_m(\epsilon)) \tag{5.5}$$

where

$$\phi_m(\epsilon) = \begin{cases} -\epsilon^2 log(\epsilon) & \text{for } m \geq 2 \\ \epsilon^2\sqrt{-log(\epsilon)} & \text{for } m = 1 \end{cases} \tag{5.6}$$

In the sequel, we are showing that $\mathcal{I}_k \sim o(\epsilon^2)$ for $k \in [\![0, m-2]\!]$ and $\mathcal{I}_{m-1} \sim O(\epsilon^2)$.

6. Estimation of $\mathcal{I}_k$ for $k \in [\![0, m-1]\!]$. The following lemma give an estimation in $o(\epsilon^2)$ of the quantity $\mathcal{I}_k$ for $k \in [\![0, m-1]\!]$.

Lemma 6.1. *Let $\mathcal{I}_k$ defined by (3.10) for $k \in [\![0, m-1]\!]$. We have*

$$\mathcal{I}_k = o(\epsilon^2), \ \forall k \in [\![0, m-2]\!] \tag{6.1a}$$

$$\mathcal{I}_{m-1} = \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^{m+1}\int_\sigma \lambda_m(y) d\sigma + o(\epsilon^2) \tag{6.1b}$$

Proof. Let $k \in [\![0, m-2]\!]$, thanks to Lemma 10.3 and Lemma 4.1 applied to u_0, we get : $\mathcal{I}_k = \epsilon^{1-k}\int_\sigma A_{m-1-k}(u_0)(\epsilon X)\left[\frac{\partial^k}{\partial n^k}(w_\epsilon(\epsilon X))\right] d\sigma \leq C\epsilon^{1-k}|w_\epsilon(\epsilon X)|_{m,B_r\setminus\overline{\sigma}}$. By using a change of variable and Lemma 10.8 we deduce that $\mathcal{I}_k \leq C\epsilon^{m-k}|w_\epsilon|_{m,\Omega_\epsilon} \leq C\epsilon^{m-k+1} = o(\epsilon^2)$ Now, let us consider $\mathcal{I}_{m-1}$; thanks to Lemma 10.8, Lemma 4.1 applied to u_0, and the jump relations (5.3), we get :

$$\begin{aligned}\mathcal{I}_{m-1} &= \epsilon^{2-m}\int_\sigma A_0(u_0(\epsilon X))\left[\frac{\partial^{m-1}(w_\epsilon(\epsilon X))}{\partial n^{m-1}}\right] d\sigma = \epsilon^{2-m}\frac{\partial^m u_0}{\partial x_2^m}(0)\int_\sigma\left[\frac{\partial^{m-1}(\epsilon^m R(X))}{\partial n^{m-1}}\right] d\sigma + \mathcal{E}_1 + \mathcal{E}_2 \\ &= \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^{m+1}\int_\sigma \lambda_m(y) d\sigma + \mathcal{E}_1 + \mathcal{E}_2\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_1 &= \epsilon^{2-m}\int_\sigma\left(A_0(u_0(\epsilon X)) - \frac{\partial^m u_0}{\partial x_2^m}(0)\right)\left[\frac{\partial^{m-1}(w_\epsilon(\epsilon X))}{\partial n^{m-1}}\right] d\sigma \\ \mathcal{E}_2 &= \epsilon^{2-m}\frac{\partial^m u_0}{\partial x_2^m}(0)\int_\sigma\left[\frac{\partial^{m-1}(e_\epsilon(\epsilon X))}{\partial n^{m-1}}\right] d\sigma\end{aligned}$$

Now let us show that $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\sim o(\epsilon^2)$. By using Lemma 10.3, a change of variable, Lemma 4.1, a change of variable again and Lemma 10.8 we get

$$\mathcal{E}_1 \leq \epsilon^{2-m}\left\|A_0(u_0(\epsilon X)) - \frac{\partial^m u_0}{\partial x_2^m}(0)\right\|_{H_{00}^{1/2}(\sigma)'} |w_\epsilon(\epsilon X))|_{m,B_r\setminus\overline{\sigma}} \leq C\epsilon^2|w_\epsilon|_{m,\Omega_\epsilon} \leq C\epsilon^3$$

Similarly we get :

$$\mathcal{E}_2 \leq \epsilon^{2-m}|e_\epsilon(\epsilon X)|_{m,B_r\setminus\overline{\sigma}} \leq C\epsilon|e_\epsilon|_{m,\Omega_\epsilon} \leq C\epsilon\phi_m(\epsilon)$$

where ϕ_m is defined in (5.6) and is such that $\phi_m(\epsilon) = o(\epsilon)$. Hence, the following estimation holds :

$$\mathcal{I}_{m-1} = \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^{m+1}\int_\sigma \lambda_m(y) d\sigma + o(\epsilon^2)$$

□

7. Computation of the topological gradient. From (3.9) and using estimations (6.1a) and (6.1b) we obtain :

$$J_\epsilon(u_\epsilon) - J_0(u_0) = \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^m \int_\sigma \lambda_m(y) d\sigma + \mathcal{J}_\epsilon + o(\epsilon^2)$$

From the expression of λ_m (5.4) and the definition of $\mathcal{J}_\epsilon$ (3.6) we have :

$$\begin{aligned}
J_\epsilon(u_\epsilon) - J_0(u_0) &= \epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0)(-1)^m \int_{-1}^{1} \frac{(-1)^{m+1} 2^{2m-1}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\partial^m v_0}{\partial x_2^m}(0)\sqrt{1-s^2}ds + \epsilon^2 \delta J + o(\epsilon^2) \\
&= -\epsilon^2 \frac{\partial^m u_0}{\partial x_2^m}(0) \frac{\partial^m v_0}{\partial x_2^m}(0) \frac{2^{2m-1}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\pi}{2} + \epsilon^2 \delta J + o(\epsilon^2) \\
&= -\epsilon^2 \pi \frac{2^{2(m-1)}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\partial^m u_0}{\partial x_2^m}(0) \frac{\partial^m v_0}{\partial x_2^m}(0) + \epsilon^2 \delta J + o(\epsilon^2)
\end{aligned}$$

Therefore, the topological gradient written in the local coordinate system of the crack is

$$\mathcal{I}(0) = -\pi \frac{2^{2(m-1)}}{(2m-1)C_{2(m-1)}^{m-1}} \frac{\partial^m u_0}{\partial x_2^m}(0) \frac{\partial^m v_0}{\partial x_2^m}(0) + \delta J \tag{7.1}$$

8. Conclusion : general expression and some examples of cost functions. From (7.1), by a change of coordinates, we easily deduce the topological gradient associated to a cost function $J(\Omega, u)$ (verifying Hypotheses 1) and to the PDE (3.3) for a domain perturbed by a straight crack of normal $\mathbf{n}$ and of center $x_0 \in \Omega$:

(8.1)
$$\mathcal{I}(x_0, \mathbf{n}) = -\pi \frac{2^{2(m-1)}}{(2m-1)C_{2(m-1)}^{m-1}} \nabla^m u_0(x_0)(\mathbf{n}, ..., \mathbf{n}) \nabla^m v_0(x_0)(\mathbf{n}, ..., \mathbf{n}) + \delta J(x_0, \mathbf{n})$$

Now we give some cost functions examples and we compute the function $\delta J(x_0, \mathbf{n})$ in these cases.

Case of semi-norms $H^p(\Omega_\epsilon)$ for $p \in [\![1, m]\!]$. Let us define

$$J_\epsilon(u) = |u|^2_{H^p(\Omega_\epsilon)} = \int_{\Omega_\epsilon} |\nabla^p u|^2 \tag{8.2}$$

with $p \in [\![1, m]\!]$.

(i) For $p = m$, by using the equation checked by u_ϵ and Lemma 10.8 applied to $u_\epsilon - u_0$, we get

$$J_\epsilon(u_\epsilon) - J_0(u_0) = L_\epsilon(u_\epsilon - u_0) + \mathcal{J}_\epsilon \tag{8.3}$$

with $L_\epsilon(u) = \int_{\Omega_\epsilon}(f - 2u_0)u$ and $\mathcal{J}_\epsilon = -\|u_\epsilon - u_0\|^2_{0,\Omega_\epsilon} \leq C\phi_m(\epsilon)^2 = o(\epsilon^2)$. We deduce that $\delta J = 0$.

(ii) For $p \in [\![1, m-1]\!]$, an integration by parts (Theorem 1.1) and Lemma 10.19 applied to $u_\epsilon - u_0$ lead to (8.3) with

$$L_\epsilon(u) = 2\int_{\Omega_\epsilon} \nabla^p u_0.\nabla^p u \quad \text{and} \quad \mathcal{J}_\epsilon = |u_\epsilon - u_0|^2_{p,\Omega_\epsilon} = o(\epsilon^2)$$

We deduce again that $\delta J = 0$.
Thanks to Theorem 1.1, we check that Hypotheses 1 are verified :

$$L_\epsilon(u) = \int_{\Omega_\epsilon} 2(-1)^p \Delta^p u_0 u + \sum_{i=0}^{p-2} \int_{\sigma_\epsilon} -2A^p_{p-1-i}(u_0)\left[\frac{\partial^i u}{\partial n^i}\right] + \int_\Gamma 2A^p_{p-1-i}(u_0)\frac{\partial^i u}{\partial n^i}$$

where the operators $A^p_i(u)$ for $i \in [\![0, p-1]\!]$ stand for the p-Neumann conditions associated to the minimization in $H^p(\Omega)$ of $\int_\Omega |\nabla^p u|^2$.

Case of norm $H^m(\Omega_\epsilon)$. Thanks to Gagliardo-Niremberg inegality (see Introduction)

$$J_\epsilon(u) = |u|^2_{H^m(\Omega_\epsilon)} + \|u\|^2_{L^2(\Omega_\epsilon)}$$

is a norm on $H^m(\Omega_\epsilon)$. Remarking that $J_\epsilon(u_\epsilon) = \int_{\Omega_\epsilon} f u_\epsilon$ where u_ϵ is solution of (3.3) we can set $L_\epsilon(u) = \int_{\Omega_\epsilon} f u$. We deduce that $v_\epsilon = -u_\epsilon$ and $\delta J = 0$. From (8.1) we deduce that $\mathcal{I}(x_0, \mathbf{n}) \geq 0$ and so the creation of a crack creates energy : $\|u_\epsilon\|^2_{H^m(\Omega_\epsilon)} \geq \|u_0\|^2_{H^m(\Omega_\epsilon)}$.

Remark 2.

(i) We check that for $p = m$, $m = 1$ and $m = 2$ we retrieve the topological gradient expressions given respectively in [3] and [5].

(ii) In imaging, for edges and fine structures detection the reasoning is the following. Let $\bar{u}$ a regularisation of an observed image. First we choose a regularization penalizing structure that we want to detect. More precisely, in our case, we choose a semi-norm $|.|^2_{H^m(\Omega)}$ with m such that $\nabla^m \bar{u}$ be high in a neighbourhood of the structure we want to detect (see Table 1 for a reasoning on transverse cut). Then we choose the pde associated to the minimization problem (1.2) using as regularization $|.|^2_{H^m(\Omega)}$. It is easy to see that the pde is sensitive to creation of crack at point where $\nabla^m \bar{u}$ is high i.e. where there is a searched structure (edge, filament,...). Once the pde is fixed, it is possible to use different cost functions from $|.|^2_{H^m(\Omega)}$ (see [2]) : typically, we can take as cost function the semi-norms $|.|^2_{H^p(\Omega)}$ for $p \in [\![0, m]\!]$ measuring the sensitivity of the pde w.r.t to creation of crack (see section 9.2).

(iii) Generally at each $x_0 \in \Omega$, we introduce the following topological indicator associated to a cost function $J_\epsilon(u) = J(\Omega_\epsilon, u)$ verifying Hypothesis 1

$$I_{mlap} = \max_{\|\mathbf{n}\|=1} |\mathcal{I}(x_0, \mathbf{n})|$$

what means that we search an orientation of the crack centered at x_0 which lead to a maximal variation of $\epsilon \mapsto J_\epsilon(u_\epsilon)$.

9. Application in Imaging. In this section we present two applications in imaging : edges detection and fine structures detection. The two first models presented here ($m = 1$ and $m = 2$) have already been introduced before [7, 9, 5], we briefly recall them. The third model ($m = 3$) is new, applied to edge detection and compared to the model ($m = 1$). All of the following models have been obtained by using Matlab code and the computation of the direct and the adjoint states (3.3) and (3.7) have been done by Fast Fourrier Transform (FFT).

9.1. Edge detection : $m = 1$ and $m = 3$.
In the both following cases, we take as cost function $J_\epsilon(u) = |u|^2_{H^m(\Omega_\epsilon)}$.

Case $m = 1$.
In this case we can extend the model to a more general model of deblurring where the observed image is $f = Ku + b$ with u is the image to recover, b a gaussian noise identically distributed and K the blur modeled by a convolution such that $K\mathbb{1} \neq 0$. Hence, the data fidelity term used in (1.2) is replaced by $\|Ku - f\|^2_{L^2(\Omega)}$. If $K\mathbb{1} \neq 0$ we can show (see [6] chapter 3) that the problem is well-posed. The direct model (3.3) and the adjoint model (3.7) write as

$$(\mathcal{P}_0)\begin{cases} -\gamma\Delta u_0 + K^\star K u_0 = K^\star f, \text{ in } \Omega \\ \partial_n u_0 = 0, \text{ on } \partial\Omega \end{cases} \tag{9.1a}$$

$$(\mathcal{Q}_0)\begin{cases} -\gamma\Delta v_0 + K^\star K v_0 = K^\star(2Ku_0 - f), \text{ in } \Omega \\ \partial_n v_0 = 0, \text{ on } \partial\Omega. \end{cases} \tag{9.1b}$$

where f is the observed image and $\gamma > 0$. Since the 2D Lebesgue measure of the crack is zero, the topological gradient is unchanged w.r.t u_0 and v_0 which are now given respectively by (9.1a) and (9.1b). Numerically we observe that $\mathcal{I}(x_0, \mathbf{n})$ is always negative on edges, hence we introduce the following topological indicator (see [11, 7])

$$I_{Lap} = \min_{\|\mathbf{n}\|=1} \mathcal{I}(x_0, \mathbf{n}) \tag{9.2}$$

We denote the edge indicator used by the famous Canny Algorithm by I_{Canny}. More precisely I_{Canny} is the norm of the Gradient of a regularization by Gaussian convolution at scale σ of the image. In Figure 4 we compare results obtained from I_{Lap} and I_{Canny}. Edges detected by I_{Lap} are thinner than for I_{Canny} and I_{Lap} is qualitatively more robust. For more details on this model and more experimentations we refer the reader to [11, 7].

Remark 3.

(i) The model presented here is extendable to semi-linear problems adapted to another kind of noise like Poisson noise, and Speckle noise (see [11]).

(ii) For the study of $J_\epsilon(u)$ equal to the semi-norm $|u|^2_{H^1(\Omega_\epsilon)}$ or to the norm $\|u\|^2_{L^2(\Omega_\epsilon)}$ we refer the reader to [2]. In particular, it is shown that taking $J_\epsilon(u) = \|u\|^2_{L^2(\Omega_\epsilon)}$ avoids edge doubling for large γ.

Case $m = 3$.
We present this model to underline Remark 2-(ii) concerning the choice of the pde w.r.t the structure we want to detect. Since on edges, $\mathcal{I}(x_0, \mathbf{n})$ is always negative, we defined the topological indicator

$$I_{Trilap}(x_0) = \min_{\|\mathbf{n}\|=1} \mathcal{I}(x_0, \mathbf{n}) \tag{9.3}$$

where $\mathcal{I}(x_0, \mathbf{n})$ is given by (8.1) for $m = 3$ and depends on the direct state (3.3) and the adjoint state (3.7) which are both solution of a 6-th order's PDE. Though the model (1.2) seems to more penalize edges for $m = 3$ than for $m = 1$, the difference between $u_\epsilon - u_0$ in the sense of the $H^m(\Omega_\epsilon)$-norm will be lower for $m = 3$ than for $m = 1$. We can see this a posteriori by looking at the asymptotic expansion of $u_\epsilon - u_0$ which is equivalent to $\epsilon^m R\left(\frac{x}{\epsilon}\right)$ (see Lemma 10.8) where R is the exterior problem

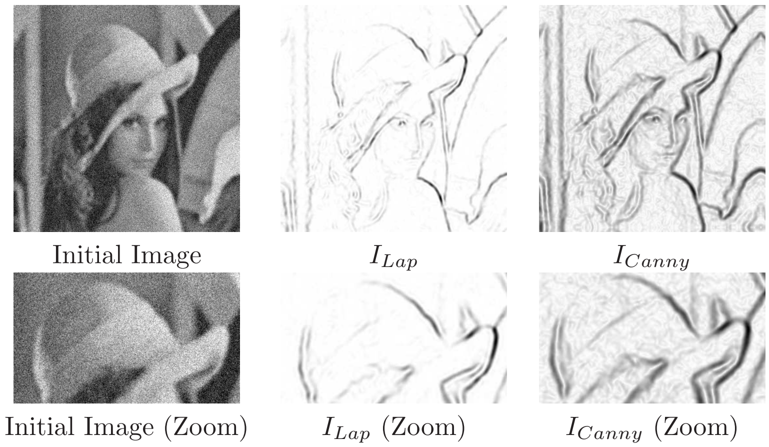


FIG. 4. *Comparison of I_{Lap} (9.2) ($\alpha = 1$) and I_{Canny} ($\sigma = 3$), on a noisy and Gaussian blurred image (PSNR=16dB, scale of blur : 3)*

(5.2) with $g_0 = -\nabla^m u_0(x_0)(\mathbf{n}, .., \mathbf{n})$ and $g_i = 0$ for $i \in [\![1, m-1]\!]$. When $m \to \infty$ the boundary condition $\nabla^m u_0(x_0)(\mathbf{n}, .., \mathbf{n})$ vanishes. Indeed the regularization for $m \geq 3$ is very violent : in the Fourrier domain the regularizing operator behave on high frequency as $\frac{1}{|\nu|^{2m}}$. This fact explains that some parts of edges associated to law contrast are missed for $m = 3$. Besides the computation of the topological gradient which is of order 3 in derivatives, lead to instabilities as we could foresee and see on Figure 5 where some oscillations appear on I_{Trilap}. After that, from Table 1, the cost function is *sensitive to changing of curvature of intensity*. Consequently, the topological gradient is sensitive to edges (inflection points) but also to variations of curvature at both side to edges. This prevision is confirmed on Figure 5 where the topological indicators I_{Lap} (9.2) and I_{Trilap} (9.3) ($m = 3$) are compared.

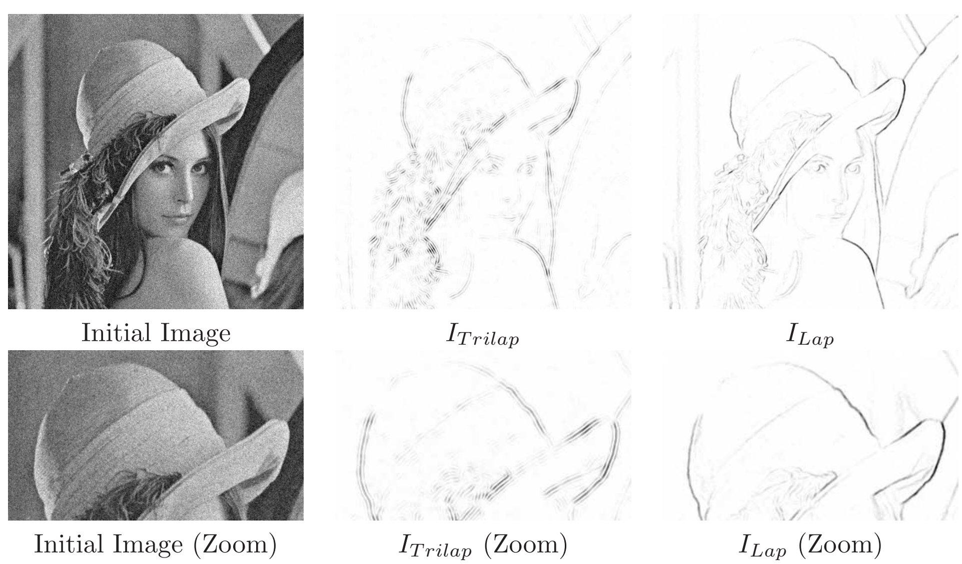


FIG. 5. *Comparison of I_{Trilap} (9.3) ($\alpha = 10^{-2}$) and I_{Lap} (9.2) ($\alpha = 1$) on a gaussian noisy image (PSNR=22dB)*

9.2. Fine structure detection : $m = 2$. We can do the same remark as for $m = 1$ concerning the extension to a deblurring model. More precisely (1.2) is well-posed if K is a convolution such that $K\mathbb{1} \neq 0$ and $K.w = K.\mathbb{1}w$ for all $w \in \mathbb{P}_1$. This is the case if Ω is a rectangle and the convolution is autoadjoint. Indeed assume that $\exists a, b > 0$ such that $\Omega = [0, a] \times [0, b]$. For $f \in L^\infty(\Omega)$ we denote by $f^\sharp$ its symmetrical and periodic extension over $\mathbb{R}^2$. For every $w \in L^\infty(\Omega)$, we define the convolution operator K of kernel $k \in L^1(\mathbb{R}^2)$ by

$$(K.w)(x) = \int_{\mathbb{R}^2} w^\sharp(x-y)k(y)dy \qquad \forall x \in \Omega$$

From this definition, if k is symmetrical we get easily that $K.x_1 = x_1$ and $K.x_2 = x_2$. By using these properties, it is straightforward to adapt the proof of Theorem 3.2.1 of [6] by splitting the solution as $u = v + P_1(u)$ where P_1 is the $L^2(\Omega)$ projection on the polynomial functions $\mathbb{P}_1$ and by using Deny-Lions inequality ([8] Lemma 5.2). The direct model (3.3) and the adjoint model (3.7) write as

$$(\mathcal{P}_0)\begin{cases} \gamma\Delta^2 u_0 + K^\star K u_0 = K^\star f, \text{ in } \Omega \\ \qquad B_1(u_0) = B_2(u_0) = 0, \text{ on } \partial\Omega \end{cases} \tag{9.4a}$$

$$(\mathcal{Q}_0)\begin{cases} \gamma\Delta^2 v_0 + K^\star K v_0 = L, \text{ in } \Omega \\ B_1(u_0) = g_1 \quad \text{and} \quad B_2(u_0) = 0, \text{ on } \partial\Omega. \end{cases} \tag{9.4b}$$

where f is the observed image, $\gamma > 0$, B_1 and B_2 are the natural boundary conditions associated to the Bilaplacian (see Remark 1, and chapter 8 of [8]), g_1 and L are data related to the cost function that we use here. In the following table we give their value according to four cost functions (the norm and the three semi-norms) :

$J_\epsilon(u)$	L	g_1
$\lvert u\rvert^2_{H^2(\Omega_\epsilon)}$	$K^\star(2Ku_0 - f)$	0
$\lvert u\rvert^2_{H^1(\Omega_\epsilon)}$	$2\Delta u_0$	$\partial_n u_0$
$\lVert u\rVert^2_{L^2(\Omega_\epsilon)}$	$-2u_0$	0
$\lVert u\rVert^2_{H^2(\Omega_\epsilon)}$	$-f$	0

The topological gradient associated to each cost function is the same w.r.t u_0 and v_0. It is given by (8.1) with $\delta J = 0$ (see section 8) and is denoted by $\mathcal{I}(x_0, \mathbf{n})$. For each cost function, we introduce the following topological indicator (see [9])

$$I_{Bilap} = \max_{\|\mathbf{n}\|=1} |\mathcal{I}(x_0, \mathbf{n})| \tag{9.5}$$

and we denote by I^0_{Bilap}, I^1_{Bilap}, I^2_{Bilap} and I^{0+1+2}_{Bilap} the topological indicator associated to respectively the L^2 norm, the H^1 semi-norm, the H^2 semi-norm and the H^2 norm. We denote by G_σ the Gaussian function of scale σ and $H_\sigma = \nabla^2 f * G_\sigma = f * \nabla^2 G_\sigma$. If we set λ_1 and λ_2 the eigenvalues of H_σ such that $0 \leq |\lambda_1| \leq |\lambda_2|$, then we define the following classical filament indicator ([16, 13]):

$$I^f_{Hes} = |\lambda_2| - |\lambda_1| \text{ (filaments indicator)} \tag{9.6}$$

In Figure 6 we compare results obtained from I^0_{Bilap}, I^1_{Bilap}, I^2_{Bilap} and I^{0+1+2}_{Bilap} and I^f_{Hes}. This figure underlines Remark 2-(ii) concerning the relation between the cost function and the pde.

For more details on this model and more experimentations we refer the reader to [9].

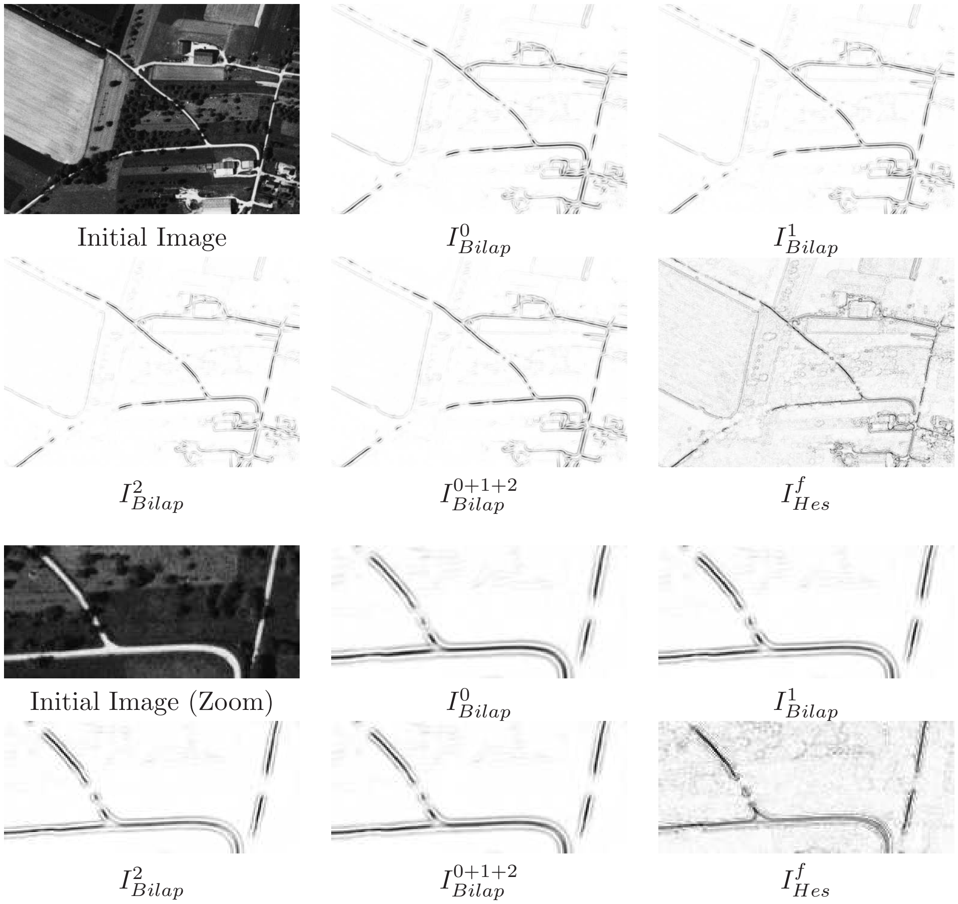


FIG. 6. *Comparison of I_{Bilap}* (9.5) *($\alpha = 10^{-2}$) for different cost functions and I_{Hes}* (9.6) *($\sigma = 5/4$) on a real image*

10. Appendix. In the following, we keep the notations and conventions described in Figure 3 and we choose $r > 0$ and ϵ small enough such that $\widetilde{\omega} \subsetneq B_r \subset \frac{\Omega}{\epsilon}$ (see Figure 3). We recall that we denote by B_r for $r > 0$ the ball of center 0 and of radius r, and $B = B_1$ is the unit ball. For a bounded domain $\omega \subset \mathbb{R}^2$, ω' stands for the unbounded domain $\mathbb{R}^2 \backslash \overline{\omega}$. Finally for a domain ω, we denote by $\mathcal{D}(\omega)$ the set of functions $C^\infty(\omega)$ with compact support in ω. The following lemma is a consequence of the Hardy inequality (see [14]) and is a generalization of the Deny-Lions inequality (see [8] Lemma 5.2).

LEMMA 10.1 (Generalization of the Deny-Lions inequality). *Let $\omega \subset B$, a regular subset of $\mathbb{R}^2$. We denote by $\mathcal{O} = \mathbb{R}^2 \backslash \overline{\omega}$ the exterior domain to ω. Let $u \in W^m(\mathcal{O})$, we have the following inequality*

$$\|u\|_{W^m(\mathcal{O})/\mathbb{P}_{m-1}} \leq C|u|_{m,\Lambda}$$

where C is a constant which depends on $\mathcal{O}$ and where $W^m(\mathcal{O})$ is the space defined in (3.1).

Proof. Let $\varphi \in C^m([0, +\infty[)$, the cut-off function equal to 0 for $0 \leq x \leq 1$ and equal to 1 for $x \geq 2$. If $\psi(x) = \varphi(|x|)$, then $u\psi \in W_0^m(B')$. On the space $W_0^m(B')$, thanks to Hardy inequality [14], we have $\|u\psi\|_{W_0^m(B')} \leq C|\psi u|_{m,B'}$. By definition of ψ, we get $\|u\|_{W^m(B_2')} \leq \|\psi u\|_{W_0^m(B')}$. We deduce that :

$$
\begin{aligned}
\|u\|_{W^m(B_2')} &\leq C|\psi u|_{m,B'} \leq C|u|_{m,B_2'} + C\|u\|_{W^m(B_2\backslash \overline{B})} \\
&\leq C|u|_{m,B_2'} + C\|u\|_{W^m(B_2\backslash\overline{\omega})}
\end{aligned} \tag{10.1}
$$

Then by using the definition of $W^m(B_2\backslash\overline{\omega})$ and by bounding from below and from above the weights, we get the equivalence between the $W^m(B_2\backslash\overline{\omega})$ and the $H^m(B_2\backslash\overline{\omega})$ norms. By considering the quotient space and thanks to Deny-Lions inequality, we get :

$$
\|u\|_{W^m(B_2\backslash\overline{\omega})/\mathbb{P}_{m-1}} \leq C|u|_{m,B_2\backslash\overline{\omega}} \tag{10.2}
$$

From (10.1) and (10.2), we have $\|u\|_{W^m(\mathcal{O})/\mathbb{P}_{m-1}} \leq C|u|_{m,\mathcal{O}}$ which ends the proof of the lemma. □

Theorem 10.2 (Gagliardo-Nirenberg inequality [1, 15]).

Let $m \geq 1$, the map $u \mapsto \left(\|u\|_{0,\Omega}^2 + |u|_{m,\Omega}^2\right)^{\frac{1}{2}}$ from $H^m(\Omega)$ to $\mathbb{R}$ is a norm on $H^m(\Omega)$ and more precisely we have

$$
\|u\|_{m,\Omega} \leq C(m,\Omega)\left(\|u\|_{0,\Omega}^2 + |u|_{m,\Omega}^2\right)^{1/2}
$$

where $C(\Omega,m)$ is a constant depending on m and Ω.

Remark 4. *The constant appearing in the Gagliardo-Nirenberg inequality depends on constants linked to the interior cone property (see [1] p 66 and pp 75-79). Let $\Omega_\epsilon = \Omega\backslash\overline{x_0+\epsilon\omega}$, where ω is either a regular open manifold or a regular sub-domain of $\mathbb{R}^2$ and ϵ small enough in order that the perturbation $\{x_0+\epsilon\omega\}$ does not touch the boundary of Ω. Then we can show that the Gagliardo-Nirenberg constant is bounded independently of ϵ when $\epsilon \to 0$.*

Lemma 10.3. *Let $k \in [\![0, m-1]\!]$, $g_k \in \left(H_{00}^{1/2+(m-1)-k}(\sigma)\right)'$ and $u \in H^m(B_r\backslash\overline{\sigma})$, we have*

$$
\left|\int_\sigma g_k\left[\frac{\partial^k u}{\partial n^k}\right]\right| d\sigma \leq \|g_k\|_{H_{00}^{1/2+m-1-k}(\sigma)'}|u|_{m,B_r\backslash\overline{\sigma}}
$$

where the spaces $H_{00}^{1/2+i}(\sigma)$ for $i \in \mathbb{N}$ are defined in section 3 and where $H_{00}^{1/2+i}(\sigma)'$ are their dual spaces.

Proof. By using the definition of the norm $\|.\|_{H_{00}^{1/2+m-1-k}(\sigma)}$, by splitting the jump $\left[\frac{\partial^k u}{\partial n^k}\right]$ and finally by using the trace Theorem on $B_r\backslash\overline{\widetilde{\omega}}$ and on $\widetilde{\omega}$, we deduce that

$$
\begin{aligned}
\int_\sigma g_k\left[\frac{\partial^k u}{\partial n^k}\right] &\leq \|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'}\left\|\left[\frac{\partial^k u}{\partial n^k}\right]\right\|_{H_{00}^{1/2+(m-1)-k}(\sigma)} \\
&= \|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'}\left\|\left[\frac{\partial^k u}{\partial n^k}\right]\right\|_{1/2+(m-1)-k,\widetilde{\sigma}} \\
&\leq C\|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'}\|u+\psi\|_{m,B_r\backslash\overline{\sigma}}
\end{aligned}
$$

where ψ is a regular function defined on $\mathbb{R}^2$. In particular, if we take $\psi \in \mathbb{P}_{m-1}$, by using the Deny-Lions inequality, we get

$$
\int_\sigma g_k\left[\frac{\partial^k u}{\partial n^k}\right] \leq C\|g_k\|_{H_{00}^{1/2+(m-1)-k}(\sigma)'}|u|_{m,B_r\backslash\overline{\sigma}}
$$

□

Lemma 10.4. *Let $u \in H^m(B_r \backslash \overline{\sigma})$ such that $\Delta^m u \in L^2(B_r \backslash \overline{\sigma})$ and $q_i \in H_{00}^{1/2+i}(\sigma)$ où $i \in [\![0, m-1]\!]$. Then we have the inequality :*

$$\sum_{i=0}^{m-1} \int_\sigma q_i A_i(u) \leq \sum_{i=0}^{m-1} \|q_i\|_{H_{00}^{1/2+i}(\sigma)} \left(|u|_{B_r \backslash \overline{\sigma}} + \|\Delta^m u\|_{B_r \backslash \overline{\sigma}} \right)$$

Proof. We extend q_i by 0 on $\widetilde{\sigma} \backslash \overline{\sigma}$ and we denote by $\widetilde{q_i} \in H^{1/2+i}(\widetilde{\sigma})$ these extensions. Let Q be a continuous extension of $(\widetilde{q_0}, \widetilde{q_1}, ..., \widetilde{q_{m-1}})$ in $\widetilde{\omega}$. By integration by parts (see Theorem 1.1), we have :

$$\widetilde{b}(Q, u) = (-1)^m \int_{\widetilde{\omega}} \Delta^m u Q + \int_{\widetilde{\sigma}} \sum_{i=0}^{m-1} A_i(u) q_i, \ \forall u \in H^m(\widetilde{\omega})$$

where

$$\widetilde{b}(u, v) = \sum_{k=0}^{m} C_m^k \int_{\widetilde{\omega}} \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}, \ \forall u, v \in H^m(\widetilde{\omega})$$

Moreover, we have :

$$\begin{aligned} \left| \int_{\widetilde{\sigma}} \sum_{i=0}^{m-1} A_i(u) q_i \right| &\leq |b(Q, u)| + \left| \int_{\widetilde{\omega}} \Delta^m u Q \right| \leq C |u|_{2,\widetilde{\omega}} |Q|_{2,\widetilde{\omega}} + \|\Delta^m u\|_{0,\widetilde{\omega}} \|Q\|_{0,\widetilde{\omega}} \\ &\leq C \left(|u|_{2,\widetilde{\omega}} + \|\Delta^m u\|_{0,\widetilde{\omega}} \right) \|Q\|_{2,\widetilde{\omega}} \end{aligned}$$

by using the continuity of the extension, the inclusion $B_r \backslash \overline{\sigma} \supset \widetilde{\omega}$ (see Figure 3), and by using the definitions of the $H_{00}^{1/2+i}(\sigma)$ norms for $i \in [\![0, m-1]\!]$, we get the result.

□

Lemma 10.5. *Let $E(x)$ given in (2.2). Let $s \in \mathbb{N}^2$, we have*

$$\frac{\partial^{|s|} E}{\partial x^s} = \begin{cases} F_s(x) + G_s log(|x|), \ \text{for } 0 \leq |s| \leq 2m-2 \\ H_s, \ \text{for } |s| \geq 2m-1 \end{cases}$$

where F_s, G_s and H_s are rational homogeneous functions of degree $2(m-1) - |s|$ and where for $s = (s_1, s_2)$, $\partial x^s = \partial x_1^{s_1} \partial x_2^{s_2}$ and $|s| = s_1 + s_2$.

Proof. E writes as the product $E(x) = C_m g(x) h(x)$ where $g(x) = (x_1^2 + x_2^2)^{m-1}$ and $h(x) = log(x_1^2 + x_2^2)$ and C_m is a constant depending on m. $g(x)$ is a polynomial homogeneous function of degree $2(m-1)$, hence $\frac{\partial^{|\alpha|} g}{\partial x^\alpha}$ for $|\alpha| \leq 2(m-1)$ is a polynomial homogeneous function of degree $2(m-1) - |\alpha|$. Similarly, $\frac{\partial h}{\partial x_i}$ is homogeneous of degree -1, and $\frac{\partial^{|\alpha|} h}{\partial x^\alpha}$ is homogeneous of degree $-|\alpha|$. Thanks to Leibniz formula, we get

$$\frac{\partial^{|s|}(gh)}{\partial x^s} \sum_{0 \leq \alpha \leq s} C_s^\alpha \frac{\partial^{|\alpha|} g}{\partial x^\alpha} \frac{\partial^{|s-\alpha|} h}{\partial x^{s-\alpha}}$$

where $C_s^\alpha = \prod C_{s_i}^{\alpha_i}$ where $\alpha \leq s \iff \alpha_i \leq s_i, \ \forall i \in \{1, 2\}$. By splitting the sum in two parts we get :

$$\frac{\partial^{|s|}(gh)}{\partial x^s} \sum_{0 \leq \alpha < s} C_s^\alpha \underbrace{\frac{\partial^{|\alpha|} g}{\partial x^\alpha} \frac{\partial^{|s-\alpha|} h}{\partial x^{s-\alpha}}}_{\text{homogeneous of degree } 2(m-1)-|s|} + \underbrace{\frac{\partial^{|s|} g}{\partial x^s}}_{\substack{\text{homogeneous of degree } 2(m-1)-|s| \\ \text{and null if } |s| \geq 2m-1}} h$$

For $|s| \geq 2m-1$, $\frac{\partial^s E}{\partial x^s}$ is a rational homogeneous function of degree $2(m-1)-|s|$ □

THEOREM 10.6. *Let $R \in W^m(\Lambda)/\mathbb{P}_{m-1}$ be the solution of the following exterior problem :*

$$(\mathcal{R}_{ext}) \begin{cases} \Delta^m R = 0, \ \text{in } \Lambda \\ A_i(R) = g_i, \ \text{on } \sigma \quad \forall i \in [\![0, m-1]\!] \end{cases} \tag{10.3}$$

with $g_i \in \left(H_{00}^{1/2+i}(\sigma)\right)'$. Then

1. *$R(x)$ is unique in $W^m(\Lambda)/\mathbb{P}_{m-1}$, and the map $(g_0, ..., g_{m-1}) \mapsto R$ is continuous from $\left(H_{00}^{1/2}(\sigma)\right)' \times ... \times \left(H_{00}^{1/2+m-1}(\sigma)\right)'$ in $W^m(\Lambda)/\mathbb{P}_{m-1}$.*
2. *$R(x)$ writes in $W^m(\Lambda)/\mathbb{P}_{m-1}$ as a sum of multi-layer potentials :*

$$R(x) = \sum_{i=0}^{m-1} \int_\sigma \lambda_i(y) A_{i,y}(E(x-y)) d\sigma_y, \ \forall x \in \Lambda$$

with $\lambda_i \in H_{00}^{1/2+i}(\sigma)$ for $i \in [\![0, m-1]\!]$.

3. *We have the following jump relations across σ :*

$$(-1)^{m+1} \left[\frac{\partial^{m-1-i} R}{\partial n^{m-1-i}}\right] = \lambda_i, \ \text{for } i \in [\![0, m-1]\!]$$

4. *The densities λ_i are given by a system of m boundary integral equations :*

$$g_j(x) = \sum_{i=0}^{m-1} \oint_\sigma \lambda_i(y) A_{j,x} A_{i,y}(E(x-y)) d\sigma_y, \ \text{for } j \in [\![0, m-1]\!]$$

where $\oint$ stands for the main Cauchy value.

5. *If $\sigma = \{(s,0), -1 < s < 1\}$, $g_0(x) = V$ where V is a constant and $g_i(x) = 0$ for $i \in [\![1, m-1]\!]$, the densities λ_i are given by*

$$\lambda_0(s) = \frac{(-1)^m 2^{2m-1}}{(2m-1) C_{2(m-1)}^{m-1}} V \sqrt{1-s^2}$$
$$\lambda_i(s) = 0, \forall i \in [\![1, m-1]\!]$$

Proof. **First point**
We keep the same notations and conventions as described in Figure 3. We introduce the two functional spaces

$$H^m(\Delta^m, \widetilde{\omega}) = \left\{u \in H^m(\widetilde{\omega}), \Delta^m u \in L^2(\widetilde{\omega})\right\}$$

$$W^m(\Delta^m, \widetilde{\omega}') = \left\{u \in W^m(\widetilde{\omega}'), (1+r^2)^{\frac{m}{2}} log(2+r^2) \Delta^m u \in L^2(\widetilde{\omega}')\right\}$$

where $\widetilde{\omega}' = \mathbb{R}^2 \backslash \overline{\widetilde{\omega}}$. We define the bilinear forms :
$\widetilde{b}(u,v)$ defined on $H^m(\widetilde{\omega}) \times H^m(\widetilde{\omega})$ by

$$\widetilde{b}(u,v) = \sum_{k=0}^{m} C_m^k \int_{\widetilde{\omega}} \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}$$

and $\widetilde{b}'(u,v)$ defined on $W^m(\widetilde{\omega}')\times W^m(\widetilde{\omega}')$ by

$$\widetilde{b}'(u,v)=\sum_{k=0}^{m}C_m^k\int_{\widetilde{\omega}'}\frac{\partial^m u}{\partial x_1^k\partial x_2^{m-k}}\frac{\partial^m v}{\partial x_1^k\partial x_2^{m-k}}$$

An integration by parts on $\widetilde{\omega}$ (see Theorem 1.1) gives :

$$\widetilde{b}(u,v)=(-1)^m\int_{\widetilde{\omega}}\Delta^m uv+\sum_{i=0}^{m-1}\int_{\partial\widetilde{\omega}}A_i(u)\frac{\partial^{m-1-i}v}{\partial n^{m-1-i}},\ \forall v\in H^m(\widetilde{\omega})$$

Similarly

$$\widetilde{b}'(u,v)=(-1)^m\int_{\widetilde{\omega}'}\Delta^m uv-\sum_{i=0}^{m-1}\int_{\partial\widetilde{\omega}}A_i(u)\frac{\partial^{m-1-i}v}{\partial n^{m-1-i}}\ \forall v\in W^m(\widetilde{\omega}')$$

Then we introduce the functional space K defined by

$$K=\left\{u\in H^m(\Delta^m,\widetilde{\omega})\times W^m(\Delta^m,\widetilde{\omega}')/\mathbb{P}_{m-1}, supp(\Delta^m u)=\sigma,[A_k(u)]_\sigma=0,\left[\frac{\partial^k u}{\partial n^k}\right]_{\widetilde{\sigma}\setminus\overline{\sigma}}=0,k\in[\![0,m-1]\!]\right\}$$

By bounding the weights used in the definition of $W^m(\Lambda)$ and using the regularity property of functions $H^m_{loc}(\Lambda)$, we can rewrite K :

$$K=\{u\in W^m(\Delta^m,\Lambda)/\mathbb{P}_{m-1}, supp(\Delta^m u)=\sigma,[A_k(u)]_\sigma=0,k\in[\![0,m-1]\!]\}$$

Therefore, the variational formulation of $(\mathcal{R}_{ext})$ writes :

$$\text{find } R\in K \text{ such that : }\quad b(R,v)=l(v),\ \forall v\in K,\qquad(\mathcal{R}_{ext})$$

where $l(v)$ and $b(u,v)$ are respectively the linear and bilinear forms on K :

$$l(v)=\sum_{i=0}^{m-1}\int_\sigma g_i\left[\frac{\partial^{m-1-i}v}{\partial n^{m-1-i}}\right]\qquad\text{and}\qquad b(u,v)=\sum_{k=0}^{m}C_m^k\int_\Lambda\frac{\partial^m u}{\partial x_1^k\partial x_2^{m-k}}\frac{\partial^m v}{\partial x_1^k\partial x_2^{m-k}}$$

The problem $(\mathcal{R}_{ext})$ has a unique solution in K. Indeed, the problem is coercive on K :

$$b(u,u)\geq|u|^2_{W^m(\Lambda)}\tag{10.4}$$

and thanks to Lemma 10.1

$$\|u\|_K=\|u\|_{W^m(\Lambda)/\mathbb{P}_{m-1}}\leq C(\Lambda)|u|_{W^m(\Lambda)}\tag{10.5}$$

which shows that $b(u,v)$ is coercive on K. Thanks to Lemma 10.3, we have

$$|l(v)|\leq C\sum_{i=0}^{m-1}\|g_i\|_{H_{00}^{1/2+i}(\sigma)'}\|v\|_{W^m(\Lambda)/\mathbb{P}_{m-1}}\qquad\forall v\in K\tag{10.6}$$

which proves that linear form $l(v)$ is continuous on K. As K is a closed sub-vector space of $W^m(\Lambda)/\mathbb{P}_{m-1}$ which is an Hilbert space, we deduce that it is an Hilbert.

Thanks to Lax-Milgram lemma, we get the existence and the uniqueness of the solution of problem $(\mathcal{R}_{ext})$. From the variational formulation of $(\mathcal{R}_{ext})$, (10.4), (10.5) and (10.6) we get the continuity of the map $(g_0, ..., g_{m-1}) \mapsto R$ for the associated topology and we define the isomorphism :

$$J_0 : \begin{array}{ccc} (g_0, g_1, ... & ..., g_{m-2}, g_{m-1}) & \longmapsto R \\ \left(H_{00}^{1/2}(\sigma)\right)' \times ... & \times \left(H_{00}^{1/2+m-1}(\sigma)\right)' & \longrightarrow K \end{array} \tag{10.7}$$

Second and third points

We consider the following problem : for $(q_0, q_1, ..., q_{m-1}) \in H_{00}^{1/2}(\sigma) \times ... \times H_{00}^{1/2+m-1}(\sigma)$

$$\text{find } Q \in K \text{ tel que } \left[\frac{\partial^{m-1-i} Q}{\partial n^{m-1-i}}\right] = q_i, \ \forall i \in [\![0, m-1]\!] \quad (\mathcal{Q}_{ext})$$

Let $u, v \in K$, we have

$$b(u, v) = (-1)^m \int_\Lambda \Delta^m u v - \sum_{i=0}^{m-1} \int_\sigma A_i(v) \left[\frac{\partial^{m-1-i} u}{\partial n^{m-1-i}}\right]$$

The variational formulation of $(\mathcal{Q}_{ext})$ is :

$$\text{find } Q \in K \text{ such that :} \qquad b(Q, v) = l'(v) \ , \forall v \in K, \qquad (\mathcal{Q}_{ext})$$

where $l'(v) = -\sum_{i=0}^{m-1} \int_\sigma q_i A_i(v)$. To show that $(\mathcal{Q}_{ext})$ is coercive we use the same reasoning as for $(\mathcal{R}_{ext})$. Thanks to Lemma 10.4, we get $l'(v) \leq C \sum_{i=0}^{m-1} \|q_i\|_{H_{00}^{1/2+i}(\sigma)} |v|_{m, B_r \setminus \overline{\sigma}}$. By using the equivalence between the norm and the semi-norm on $W^2(\Lambda)/\mathbb{P}_1$, we deduce that the linear form $l'(v)$ is continuous on K. Thanks to Lax-Milgram lemma, we deduce that there exists an unique solution Q of $(\mathcal{Q}_{ext})$. From the variational formulation of $(\mathcal{Q}_{ext})$, we show that the map $(q_0, ...q_{m-1}) \mapsto Q$ is continuous for the associated topology and we define the isomorphism :

$$J_1 : \begin{array}{ccc} (q_0, q_1, ... & ..., q_{m-2}, q_{m-1}) & \longmapsto Q \\ H_{00}^{1/2}(\sigma) \times ... & \times H_{00}^{1/2+m-1}(\sigma) & \longrightarrow K \end{array} \tag{10.8}$$

We denote by $J = J_1^{-1} \circ J_0$ with J_0 defined in (10.7), the isomorphism :

$$J : \begin{array}{ccccc} (g_0, g_1, ... & ..., g_{m-2}, g_{m-1}) & \longmapsto & (q_0, q_1, ... & ..., q_{m-2}, q_{m-1}) \\ \left(H_{00}^{1/2}(\sigma)\right)' \times ... & \times \left(H_{00}^{1/2+m-1}(\sigma)\right)' & \longrightarrow & H_{00}^{1/2}(\sigma) \times ... & \times H_{00}^{1/2+m-1}(\sigma) \end{array}$$

J is the map corresponding to the Neumann to Dirichlet problem $(A_0(u), ..., A_{m-1}(u)) \to \left(\left[\frac{\partial^{m-1} u}{\partial n^{m-1}}\right], ..., [u]\right)$ where $u \in K$. Let $\bar{u}$ defined by :

$$\bar{u}(x) = \sum_{i=0}^{m-1} \int_\sigma \lambda_i(y) A_{i,y}(E(x-y)) d\sigma(y) = \sum_{i=0}^{m-1} \int_{\partial\widetilde{\omega}} \widetilde{\lambda}_i(y) A_{i,y}(E(x-y)) d\sigma(y)$$

where $\lambda_i \in H_{00}^{1/2+i}(\sigma)$ and where $\widetilde{\lambda}_i \in H^{1/2+i}(\widetilde{\sigma})$ is the canonical extension of λ_i by zeros on $\widetilde{\sigma} \setminus \overline{\sigma}$.

Let us show that $\Delta^m \bar{u} = 0$ on Λ. For $y \in \sigma$, the functions $A_{i,y}(E(.-y))$ belong to $C^\infty(\Lambda)$. Moreover $\Delta^m_x A_{i,y}(E(x-y)) = A_{i,y}(\Delta^m(E(x-y))) = 0$. By using the regularity of the functions $A_{i,y}E(.-y)$ and the fact that their m-th Laplacian is null and so uniformly bounded with respect to x, we can switch the integral symbol and the Δ^m operator, which leads to : $\Delta^m \bar{u}(x) = 0$, $\forall x \in \Lambda$. Thanks to a Taylor expansion of $E(.-y)$ at point $x \in \Lambda$ for $|x| \to \infty$, and by using the $A_i(u)$ expressions for a straight crack (Theorem 1.1) and Lemma 10.5 we have $\bar{u}(x) = O\left(|x|^{m-2} log(|x|)\right)$. We deduce that $\frac{\bar{u}}{(1+r^2)^{\frac{m}{2}} log(2+r^2)} \in L^2(\Lambda)$. Similarly, by $k \in [\![1, m]\!]$ derivations of $\bar{u}$, we get that $\frac{\nabla^k \bar{u}}{(1+r^2)^{\frac{m-k}{2}} log(2+r^2)} \in L^2(\Lambda)$. We conclude that $\bar{u} \in W^m(\Lambda)$. By considering $\bar{u}$ as an element of $W^m(\Lambda)/\mathbb{P}_1$ we get $\bar{u} \in K$. Thanks to Lemma 10.10, by considering the definition of $\widetilde{\omega}$ and the definitions of $\widetilde{\lambda_i}$, we have $\forall i \in [\![0, m-1]\!]$:

$$\frac{\partial^{m-1-i} \bar{u}^{\pm}}{\partial n^{m-1-i}}(x) = \frac{\pm(-1)^{m+1}}{2} \lambda_i(x) + \sum_{j=0}^{m-1} \int_\sigma \lambda_j(y) \frac{\partial^{m-1-i}}{\partial n^{m-1-i}} \left(A_{j,y}(E(x-y)\right) d\sigma_y, \ \forall x \in \sigma$$

We deduce the jump relations across σ : $\lambda_i = (-1)^{m+1} \left[\frac{\partial^{m-1-i} \bar{u}}{\partial n^{m-1-i}}\right]$, for $i \in [\![0, m-1]\!]$. By setting $\lambda_i = (-1)^{m+1} \left[\frac{\partial^{m-1-i} R}{\partial n^{m-1-i}}\right]$, as J_1 is an isomorphism we get $\bar{u} = R$ which ends the proof of points 2 and 3.

Fourth point

By applying $A_{i,x}$ to $\bar{u}$ for $i \in [\![0, m-1]\!]$, by doing $x \to \sigma$, thanks to Lemma 10.5 we get $A_{i,x} A_{j,y}(E(x-y)) = O\left(\frac{1}{|x-y|^{2+i+j}}\right)$, where $i, j \geq m$. By using the regularity of such potentials across σ, (see Lemma 10.10), we obtain the m boundary integral equations which define J^{-1} :

$$g_i(x) = \sum_{j=0}^{m-1} \oint_\sigma \lambda_j(y) A_{i,x} A_{j,y}(E(x-y)) d\sigma_y, \ \forall i \in [\![0, m-1]\!] \tag{10.9}$$

where $\oint$ stands for the main Cauchy value.

Last point

In the straight crack case $\sigma = \{(s, 0), -1 < s < 1\}$, by setting $x = (s, 0)$ and $y = (t, 0)$ and by using the expression of A_i for a straight crack (Theorem 1.1) and Lemma 10.5, we have :

$$A_{i,x} A_{j,y}(E(x-y)) = \frac{a_{i,j}}{(s-t)^{2+i+j}} \tag{10.10}$$

where a_{ij} are some constants. (10.9) rewrites as :

$$g_i(s) = \sum_{j=0}^{m-1} a_{ij} \oint_{-1}^{1} \frac{\lambda_j(t)}{(s-t)^{2+i+j}}, \ \text{pour } i \in [\![0, m-1]\!] \tag{10.11}$$

We set $f_j(s) = \frac{1}{\pi}\int_{-1}^{1} \frac{\lambda_j(t)}{s-t}dt$, for $-1 < s < 1$ and $j \in [\![0, m-1]\!]$. By derivation of f_j we show for $n \geq 0$ that $\frac{d^n f_j}{ds^n} = \frac{(-1)^n n!}{\pi}\int_{-1}^{1} \frac{\lambda_j(t)}{(s-t)^{n+1}}dt$. Denoting $\frac{d^n f_j}{ds^n} = f_j^{(n)}$, (10.11) rewrites as :

$$g_i(s) = \sum_{j=0}^{m-1} b_{ij} f_j^{(i+j+1)}(s), \quad \text{for } i \in [\![0, m-1]\!] \tag{10.12}$$

where we set

$$b_{ij} = \frac{a_{ij}(-1)^{i+j+1}\pi}{(i+j+1)!} \tag{10.13}$$

We rewrite (10.12) with the expressions of the g_i's given at the fifth point of the theorem :

$$(\mathcal{S}) \begin{cases} V = b_{0,0} f_0^{(1)} + b_{0,1} f_1^{(2)} + ... & + b_{0,m-1} f_{m-1}^{(m)} \\ 0 = b_{1,0} f_0^{(2)} + b_{1,1} f_1^{(3)} + ... & + b_{1,m-1} f_{m-1}^{(m+1)} \\ \vdots & \vdots \\ 0 = b_{m-1,0} f_0^{(m)} + b_{m-1,1} f_1^{(m+1)} + ... & + b_{m-1,m-1} f_{m-1}^{(2m-1)} \end{cases} \tag{10.14}$$

To solve $(\mathcal{S})$, we integrate i times the i-th row by taking as constants of integration 0 for the first $i-2$ integrations and $\frac{V b_{i,0}}{b_{0,0}}$ for the $i-1$-th. The last constant is set to 0. In the sequel we will check that the constant $b_{0,0}$ is not null. The system $(\mathcal{S})$ becomes :

$$(\mathcal{S}') \begin{cases} V s = b_{0,0} f_0 + b_{0,1} f_1^{(1)} + ... & + b_{0,m-1} f_{m-1}^{(m-1)} \\ \dfrac{V b_{1,0} s}{b_{0,0}} = b_{1,0} f_0 + b_{1,m} f_1^{(1)} + ... & + b_{1,m-1} f_{m-1}^{(m-1)} \\ \vdots & \vdots \\ \dfrac{V b_{2m-1,m} s}{b_{0,0}} = b_{m-1,0} f_0 + b_{m-1,1} f_1^{(1)} + ... & + b_{m-1,m-1} f_{m-1}^{(m-1)} \end{cases} \tag{10.15}$$

The unknowns of $(\mathcal{S}')$ are the $f_i^{(i)}$ for $i \in [\![0, m-1]\!]$. A trivial solution is

$$f_0 = \frac{V s}{b_{0,0}} \quad \text{and} \quad f_i = 0, \text{ for } i \in [\![1, m-1]\!]$$

We get m uncoupled boundary integral equations. To solve the first one, we use [12] and we get $\lambda_m(s) = \frac{V}{b_{0,0}}\sqrt{1-s^2}$ for $-1 < s < 1$. For more details we refer the reader to [10]. The other equations have the trivial solution $\lambda_i = 0$ for $i \in [\![1, m-1]\!]$. To sump up the λ_i are given by :

$$\begin{cases} \lambda_0 = \dfrac{V}{b_{0,0}}\sqrt{1-s^2}, \quad -1 < s < 1 \\ \lambda_i = 0, \text{ for } -1 < s < 1 \text{ and } i \in [\![1, m-1]\!] \end{cases}$$

We check that $\lambda_i \in H_{00}^{1/2+i}(\sigma)$ and by using that J is injective, this solution is unique. From (10.13), we have $b_{0,0} = -\pi a_{0,0}$ and $a_{0,0}$ is given by (see (10.10)) :

$$A_{0,x} A_{0,y}(E(x-y)) = \frac{a_{0,0}}{(x_1 - y_1)^2}, \text{ for } x, y \in \sigma$$

From Remark 1, we can explicit the operator A_0 and we get

$$(-1)^m \frac{\partial^{2m}}{\partial x_2^{2m}} E(x-y) = \frac{a_{m,m}}{(x_1-y_1)^2}, \text{ for } x,y \in \sigma \tag{10.16}$$

We show (see Lemma 10.9 with $n = m-1$) that $\frac{\partial^{2m}}{\partial x_2^{2m}} \left(|x|^{2(m-1)} log(|x|^2)\right)_{|x_2=0} = \frac{2(2m-1)!}{x_1^2}$. Therefore, from (2.2) and (10.16) we deduce

$$a_{0,0} = \frac{(-1)^{m+1}(2m-1)!}{2^{2m-1}\pi((m-1)!)^2} \tag{10.17}$$

The expressions of $b_{0,0}$ and λ_0 are

$$b_{0,0} = \frac{(-1)^m(2m-1)}{2^{2m-1}} C_{2(m-1)}^{m-1} \quad \text{and} \quad \lambda_0(s) = \frac{(-1)^m 2^{2m-1}}{(2m-1)C_{2(m-1)}^{m-1}} V\sqrt{1-s^2}$$

which ends the proof of the last point and of the theorem.
□

Lemma 10.7. *Let $R(x)$ the solution of the problem* (10.3).
For $m \geq 1$, by using the convention $[\![0,-1]\!] = \emptyset$, we have the following estimations for $|x| \to \infty$ and $\epsilon \to 0$:

$$|\nabla^k R(x)| \leq C|x|^{m-2-k} log(|x|), \; k \in [\![0, m-2]\!]$$
$$|\nabla^{m-1} R(x)| \leq \frac{C}{|x|} \quad , \quad |\nabla^m R(x)| \leq \frac{C}{|x|^2}$$
$$\left\|\nabla^k R\left(\frac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} = O\left(-\epsilon^{-(m-2-k)} log(\epsilon)\right), \; for \; k \in [\![0, m-2]\!]$$
$$\left\|\nabla^{m-1} R\left(\frac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} = O\left(\epsilon\sqrt{-log(\epsilon)}\right) \quad , \quad \left\|\nabla^m R\left(\frac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} = O(\epsilon)$$

Proof. From Theorem 10.6, $R(x)$ writes

$$R(x) = \sum_{i=0}^{m-1} \int_\sigma \lambda_i(y) A_{i,y}(E(x-y)) d\sigma_y \tag{10.18}$$

Then a Taylor expansion at x for $|x| \longrightarrow \infty$ and standard computation lead to the result. For more details se [10]. □

Lemma 10.8. *Let w_ϵ the solution of $(\mathcal{Q}_\epsilon^c)$ given in* (5.1) *for $m \geq 1$, and R the solution of $(\mathcal{R}_{ext})$ given in* (5.2)*, we have the following asymptotic expansion*

$$w_\epsilon = \epsilon^m R\left(\frac{x}{\epsilon}\right) + e_\epsilon \tag{10.19}$$

with

$$\|e_\epsilon\|_{m,\Omega_\epsilon} = O(\phi_m(\epsilon)) \quad , \quad |w_\epsilon|_{k,\Omega_\epsilon} = O(-\epsilon^2 log(\epsilon)) \quad \forall k \in [\![0, m-2]\!]$$
$$|w_\epsilon|_{m-1,\Omega_\epsilon} = O(\epsilon^2\sqrt{-log(\epsilon)}) \quad , \quad |w_\epsilon|_{m,\Omega_\epsilon} = O(\epsilon)$$

where

$$\phi_m(\epsilon) = \begin{cases} -\epsilon^2 log(\epsilon) & for \; m \geq 2 \\ \epsilon^2\sqrt{-log(\epsilon)} & for \; m = 1 \end{cases}$$

Proof. e_ϵ is defined by :

$$(10.20)\qquad (\mathcal{E}_\epsilon)\begin{cases} (-1)^m\Delta^m e_\epsilon + e_\epsilon = -\epsilon^m R\left(\dfrac{x}{\epsilon}\right), \text{ in } \Omega_\epsilon \\ A_0(e_\epsilon) = \varphi_0(x) = O(|x|), \text{ on } \sigma_\epsilon \\ A_i(e_\epsilon) = -A_i(v_0) - B_i, \text{ on } \sigma_\epsilon \text{ for } i \in [\![1, m-1]\!] \\ A_i(e_\epsilon) = \psi_i^\epsilon(x) = O\left(\epsilon^2\right) \text{ on } \Gamma \text{ for } i \in [\![0, m-1]\!] \end{cases}$$

where $\varphi_0(x) = -A_0(v_0)(x) + A_0(v_0)(0) = O(|x|)$ (see (4.1a)), $\psi_i^\epsilon(x) = -\epsilon^m A_0\left(R\left(\frac{x}{\epsilon}\right)\right)$, $A_i(v_0)(x) = O(1)$ on σ_ϵ and B_i is defined in (3.4).

Proof of the estimation $\psi_i^\epsilon(x) = O\left(\epsilon^2\right)$

By using Theorem 1.1, the $A_j(u)$ for $j \in [\![0, m-1]\!]$ writes as

$$(10.21)\qquad A_{j,y}(u) = \sum_{m\le |p| \le j+m} \varphi_p^j(y)\frac{\partial^{|p|}u}{\partial y^p}$$

From (10.21) and thanks to Lemma 10.5 for $x \in \Gamma$ and $y \in \sigma$ we get

$$\begin{aligned} A_{i,x}A_{j,y}\left(E\left(\frac{x}{\epsilon} - y\right)\right) &= \sum_{\substack{m\le |p|\le i+m \\ m\le |q|\le j+m}} \varphi_p^i(x)\varphi_q^j(y)\frac{\partial^{|p|+|q|}}{\partial x^p \partial x^q}\left(E\left(\frac{x}{\epsilon} - y\right)\right) \\ &= \sum_{\substack{m\le |p|\le i+m \\ m\le |q|\le j+m}} \varphi_p^i(x)\varphi_q^j(y)\epsilon^{|q|-2m+2}G_{p+q}\left(x - \epsilon y\right) = O\left(\epsilon^{2-m}\right) \end{aligned}$$

where G_k is an homogeneous function of degree $2(m-1) - |k|$ and by using the fact that $G_{p+q}\left(x - \epsilon y\right) = O\left(1\right)$ when $\epsilon \to 0$ for $x, y \in \Gamma$.

Splitting of $\|e_\epsilon\|_{m,\Omega_\epsilon}$

We split e_ϵ into the sum $e_\epsilon = e_\epsilon^1 + e_\epsilon^2$ with $e_\epsilon^1 \in H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}$ solution of the following problem

$$(\mathcal{E}_\epsilon^1)\begin{cases} (-1)^m\Delta^m e_\epsilon^1 = 0, \text{ in } \Omega_\epsilon \\ A_0(e_\epsilon^1) = \varphi_0(x) = O(|x|), \text{ on } \sigma_\epsilon \\ A_i(e_\epsilon^1) = -A_i(v_0) - B_i, \text{ on } \sigma_\epsilon, \text{ for } i \in [\![1, m-1]\!] \\ A_i(e_\epsilon^1) = 0 \text{ on } \Gamma, \text{ for } i \in [\![1, m-1]\!] \end{cases}$$

and $e_\epsilon^2 \in H^m(\Omega_\epsilon)$ solution of the problem :

$$(\mathcal{E}_\epsilon^2)\begin{cases} (-1)^m\Delta^m e_\epsilon^2 + e_\epsilon^2 = -e_\epsilon^1 - \epsilon^m R\left(\dfrac{x}{\epsilon}\right), \text{ in } \Omega_\epsilon \\ A_i(e_\epsilon^2) = 0, \text{ on } \sigma_\epsilon, \text{ for } i \in [\![0, m-1]\!] \\ A_i(e_\epsilon^2) = \psi_i^\epsilon(x), \text{ on } \Gamma \end{cases}$$

Estimation of $\|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_1}$

The variational formulation of $(\mathcal{E}_\epsilon^1)$ is : find $e_\epsilon^1 \in H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}$ such that

$$b_\epsilon(e_\epsilon^1, v) = -\int_{\sigma_\epsilon} \varphi_0 \left[\frac{\partial^{m-1} v}{\partial n^{m-1}}\right] + \sum_{k=1}^{m-1} \int_{\sigma_\epsilon} (A_k(v_0) + B_k) \left[\frac{\partial^{m-1-k} v}{\partial n^{m-1-k}}\right], \ \forall v \in H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}$$

where $b_\epsilon(u,v) = \sum_{k=0}^m C_m^k \int_{\Omega_\epsilon} \frac{\partial^m u}{\partial x_1^k \partial x_2^{m-k}} \frac{\partial^m v}{\partial x_1^k \partial x_2^{m-k}}$. We take as function test $v = e_\epsilon^1$ and we set for $k \in [\![1, m-1]\!]$, $\mathcal{J}_k = \int_{\sigma_\epsilon} (A_k(v_0) + B_k) \left[\frac{\partial^{m-1-k} e_\epsilon^1}{\partial n^{m-1-k}}\right]$, and $\mathcal{J}_0 = \displaystyle\int_{\sigma_\epsilon} \varphi_0 \left[\frac{\partial^{m-1} e_\epsilon^1}{\partial n^{m-1}}\right]$.
By using Lemma 10.3, the fact that $b_\epsilon(u,v)$ is coercive and the Deny-Lions inequality we get

$$\|e_\epsilon^1\|^2_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} \le C|e_\epsilon^1|_{m,\Omega_\epsilon} \le \sum_{k=0}^{m-1} \mathcal{J}_k \tag{10.22}$$

Let $k \in [\![1, m-1]\!]$, thanks to a change of variable, Lemma 10.3, Lemma 4.1 and a change of variable again we have

$$\mathcal{J}_k = \epsilon \int_\sigma (A_k(v_0)(\epsilon X) + B_k(\epsilon X)) \left[\frac{\partial^{m-1-k} e_\epsilon^1(\epsilon X)}{\partial n^{m-1-k}}\right] d\sigma \le C\epsilon^{1-(m-1-k)} |e_\epsilon^1(\epsilon X)|_{m, B_r \setminus \overline{\sigma}} \le C\epsilon^{k+1} |e_\epsilon^1|_{m,\Omega_\epsilon}$$

Since $k \ge 1$, we deduce that $\mathcal{J}_k \le C\epsilon^2 |e_\epsilon^1|_{m,\Omega_\epsilon}$ Similarly by using the estimation $\varphi_0(\epsilon X) = O(\epsilon)$, we have

$$\begin{aligned} \mathcal{J}_0 &= \epsilon \int_\sigma \varphi_0(\epsilon X) \left[\frac{\partial^{m-1} e_\epsilon^1}{\partial n^{m-1}}\right] \le C\epsilon \|\varphi_0(\epsilon X)\|_{H_{00}^{1/2}(\sigma)'} \epsilon^{-(m-1)} |e_\epsilon^1(\epsilon X)|_{m, B_r \setminus \overline{\sigma}} \\ &\le C\epsilon^{2-(m-1)} |e_\epsilon^1(\epsilon X)|_{m, B_r\setminus\overline{\sigma}} \le C\epsilon^2 |e_\epsilon^1|_{m,\Omega_\epsilon} \end{aligned} \tag{10.23}$$

From (10.22) and by using above estimations, we get

$$\|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} \le C\epsilon^2 \tag{10.24}$$

Estimation of $\|e_\epsilon^2\|_{m,\Omega_\epsilon}$

The variational formulation of $(\mathcal{E}_\epsilon^2)$ is : find $e_\epsilon^2 \in H^m(\Omega_\epsilon)$ such that

$$a_\epsilon(e_\epsilon^2, v) = \int_{\Omega_\epsilon} \left(-\epsilon^m R\left(\frac{x}{\epsilon}\right) - e_\epsilon^1\right) v + \sum_{i=0}^{m-1} \int_\Gamma \psi_i^\epsilon \frac{\partial^{m-1-i} v}{\partial n^{m-1-i}} \qquad \forall v \in H^m(\Omega_\epsilon)$$

We take as test function $v = e_\epsilon^2$ and we set

$$\mathcal{K}_\epsilon = -\int_{\Omega_\epsilon} \left(\epsilon^m R\left(\frac{x}{\epsilon}\right) + e_\epsilon^1\right) e_\epsilon^2 \qquad \mathcal{L}_\epsilon = \sum_{i=0}^{m-1} \int_\Gamma \psi_i^\epsilon \frac{\partial^{m-1-i} e_\epsilon^2}{\partial n^{m-1-i}}$$

By using the definition of $a_\epsilon(u,v)$ (1.4), we deduce that

$$\|e_\epsilon^2\|^2_{m,\Omega_\epsilon} = \mathcal{K}_\epsilon + \mathcal{L}_\epsilon \tag{10.25}$$

Lemma 10.7 and the estimation $\|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} = O(\epsilon^2)$ give

$$\mathcal{K}_\epsilon \leq C\phi_m(\epsilon)\|e_\epsilon^2\|_{0,\Omega_\epsilon} \tag{10.26}$$

Thanks to the estimation $\psi_i^\epsilon(x) = O(\epsilon^2)$ and to the trace Theorem on $\Omega\backslash\overline{B}$, we get $\mathcal{L}_\epsilon \leq C\epsilon^2\|e_\epsilon^2\|_{m,\Omega\backslash\overline{B}} \leq C\epsilon^2\|e_\epsilon^2\|_{m,\Omega_\epsilon} \leq C\epsilon^2$. From this last inequality (10.25) and (10.26) we have

$$\|e_\epsilon^2\|_{m,\Omega_\epsilon} \leq C\phi_m(\epsilon) \tag{10.27}$$

By using (10.24) and (10.27) we get the first estimation :

$$\|e_\epsilon\|_{m,\Omega_\epsilon} \leq \|e_\epsilon^1\|_{H^m(\Omega_\epsilon)/\mathbb{P}_{m-1}} + \|e_\epsilon^2\|_{m,\Omega_\epsilon} \leq C\phi_m(\epsilon)$$

Finally, by differentiating m-times (10.19) and by using Lemma 10.7 we obtain the second estimation : $|w_\epsilon|_{m,\Omega_\epsilon} \leq \left\|\nabla^m R\left(\dfrac{x}{\epsilon}\right)\right\|_{0,\Omega_\epsilon} + |e_\epsilon|_{m,\Omega_\epsilon} \leq C\epsilon$ ∎

Lemma 10.9.
We have the following equality * :

$$\frac{\partial^{2n+2}}{\partial x_2^{2n+2}}\left(|x|^{2n}log(|x|^2)\right)_{|x_2=0} = \frac{2(2n+1)!}{x_1^2} \quad \textit{for} \quad x_1 \neq 0$$

Proof. See [10] for a complete proof. ∎

Lemma 10.10. *Let Ω be a bounded domain of $\mathbb{R}^2$, $f \in H^\alpha(\partial\Omega)$ with $\alpha > 0$, and let $V_k(f) : H^\alpha(\partial\Omega) \to H^{\alpha+2m-k}(\partial\Omega)$ for $k \in [\![1, 2m]\!]$ the multi-layer potentials :*

$$V_k(f) = \begin{cases} \displaystyle\int_{\partial\Omega} f(y)\frac{\partial^{k-1}(E(x-y))}{\partial n_y^{k-1}}d\sigma(y) \text{ for } 1 \leq k \leq m \\ \displaystyle\int_{\partial\Omega} f(y)A_{k-m-1,y}(E(x-y))d\sigma(y) \text{ for } m+1 \leq k \leq 2m \end{cases}$$

where $E(x)$ is the fundamental solution of $-\Delta^m$ given in (2.2). We denote

$$V_k^-(f)(x) = V_k(f)(x) \text{ for } x \in \Omega$$
$$V_k^+(f)(x) = V_k(f)(x) \text{ for } x \in \Omega^c$$

and for $x \in \partial\Omega$:

$$V_k^+(f)(x) = \lim_{\substack{x\to y\\ x\in\Omega^c}} V_k^+(f)(y), \qquad V_k^-(f)(x) = \lim_{\substack{x\to y\\ x\in\Omega}} V_k^-(f)(y)$$

We have the following jump relations across $\partial\Omega$ and for $x \in \partial\Omega$:

$$(\mathcal{R}_1)\begin{cases} \frac{\partial^k}{\partial n^k}V_j^+(f)(x) = \frac{\partial^k}{\partial n^k}V_j^-(f)(x) = \frac{\partial^k}{\partial n^k}V_j(f)(x), \text{ in } H^{\alpha+2m-k-j}(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \wedge j \neq 2m-k \\ \frac{\partial^k}{\partial n^k}V_{2m-k}^\pm(f)(x) = \pm\frac{(-1)^{m+1}}{2}f(x) + V_{2m-k}(f), \text{ in } H^\alpha(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \end{cases}$$

$$(\mathcal{R}_2)\begin{cases} A_kV_j^\pm(f)(x) = A_kV_j(f)(x), \text{ dans } H^{\alpha+m-k-j}(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \wedge j \neq m-k \\ A_kV_{m-k}^\pm(f)(x) = \pm\dfrac{(-1)^m}{2}f(x) + A_kV_{m-k}(f)(x), \text{ dans } H^\alpha(\partial\Omega) \text{ for } k \in [\![0, m-1]\!] \end{cases}$$

*Thanks to Arthur Vavasseur for the proof of this Lemma *where we have denoted respectively in $H^{\alpha+2m-k-j}(\partial\Omega)$ and $H^{\alpha+m-k-j}(\partial\Omega)$ the following potentials (when kernels are singular for $x \in \partial\Omega$, these boundary integrals are defined in the sense of the main Cauchy value) for $k \in [\![0, m-1]\!]$ and $j \in [\![1, 2m]\!]$:*

$$\frac{\partial^k}{\partial n_x^k} V_j(f)(x) = \begin{cases} \displaystyle\int_{\partial\Omega} \frac{\partial^k}{\partial n^k} \frac{\partial^{j-1}}{\partial n_y^{j-1}} (E(x-y)) f(y) d\sigma(y) \text{ for } 1 \leq j \leq m \\ \displaystyle\int_{\partial\Omega} \frac{\partial^k}{\partial n_x^k} A_{j-m-1,y}(E(x-y)) f(y) d\sigma(y) \text{ for } m+1 \leq j \leq 2m \end{cases}$$

and

$$A_k V_j(f)(x) = \begin{cases} \displaystyle\int_{\partial\Omega} A_{k,x} \frac{\partial^{j-1}}{\partial n_y^{j-1}} (E(x-y)) f(y) d\sigma(y) \text{ for } 1 \leq j \leq m \\ \displaystyle\int_{\partial\Omega} A_{k,x} A_{j-m-1,y}(E(x-y)) f(y) d\sigma(y) \text{ for } m+1 \leq j \leq 2m \end{cases}$$

Proof. The proof of this lemma is in part heuristic and is inspired from [[8] pp 384-388]. We show the jump properties for particular densities which are in the following space (the image of the Calderòn projector associated to the operator Δ^m) :

$$\mathcal{P} = \left\{ (u_{|\partial\Omega}, \partial_n u_{|\partial\Omega}, ..., \partial_n^{m-1} u_{|\partial\Omega}, A_0 u_{|\partial\Omega}, ..., A_{m-1} u_{|\partial\Omega}), u \in H^m(\Omega) \text{ and } \Delta^m u = 0 \right\}$$

Moreover, we use in this proof the kernel expressions in the case of a straight manifold to discuss about the singularity of the kernels. From Theorem 1.1 we have the following equality :

(10.28)

$$\begin{aligned} u(x) &= \int_\Omega \delta(x-y) u \\ &= -\int_\Omega \Delta^m u E(x-y) dy + \sum_{i=0}^{m-1} \int_{\partial\Omega} \left\{ A_{i,y}(E(x-y)) \frac{\partial^{m-1-i} u(y)}{\partial n^{m-1-i}} - A_{i,y}(u) \frac{\partial^{m-1-i} E(x-y)}{\partial n^{m-1-i}} \right\} d\sigma(y) \end{aligned}$$

Let $w \in H^{2m}(\Omega)$ such that $\Delta^m w = 0$ in Ω and $F_k \in H^{\alpha+2m-k}(\partial\Omega)$ for $1 \leq k \leq 2m$ defined by

$$\begin{cases} F_k = -A_{m-k}(w), \text{ on } \partial\Omega \text{ for } 1 \leq k \leq m \\ F_{2m-k} = \dfrac{\partial^k w}{\partial n^k}, \text{ on } \partial\Omega \text{ for } 0 \leq k \leq m-1 \end{cases} \tag{10.29}$$

For $x \in \Omega$, thanks to (10.28), we have

$$w(x) = \sum_{i=1}^{2m} V_i^-(F_i)(x) \tag{10.30}$$

First, we extend the vectors fields $\mathbf{n}(x)$ and $\boldsymbol{\tau}(x)$ in C^∞-vectors fields on a neighbourhood of $\partial\Omega$ that we denote respectively $\widetilde{\mathbf{n}}(x)$ and $\widetilde{\boldsymbol{\tau}}(x)$. Then we extend the operators $f \mapsto \frac{\partial^{2m-1-k} f}{\partial n^{2m-1-k}}$ and $f \mapsto A_k(f)$ for $k \in [\![m, 2m-1]\!]$ by using the vectors fields $\widetilde{\mathbf{n}}(x)$ and $\widetilde{\boldsymbol{\tau}}(x)$.

(i) **Proof of the relations** $(\mathcal{R}_1)$

Consider $0 \leq k \leq m-1$, from (10.30), we get

$$\frac{\partial^k w}{\partial \widetilde{n}^k} = (-1)^m \sum_{i=1}^{2m} \frac{\partial^k}{\partial \widetilde{n}^k} V_i^-(F_i)(x), \qquad x \in \Omega$$

Letting x tend nontangentially to $\partial\Omega$ in (10.28), we only have half of the contribution of the Dirac function $\delta(x-y)$ at $y = x \in \partial\Omega$ so

$$\frac{1}{2}\frac{\partial^k w}{\partial n^k}(x) = (-1)^m \sum_{i=1}^{2m} \frac{\partial^k}{\partial n^k} V_i(F_i)(x), \qquad x \in \partial\Omega \tag{10.31}$$

Thanks to Lemma 10.5 and Theorem 1.1, we get the following singularities estimations in the case of a straight manifold

$$\text{for } k+j < 2m \qquad \frac{\partial^k}{\partial n^k} V_j^-(x) = O\left(|x-y|^{2m-1-(k+j)} log(|x-y|\right)$$
$$\text{for } k+j \geq 2m \qquad \frac{\partial^k}{\partial n^k} V_j^-(x) = O\left(\frac{1}{|x-y|^{k+j+1-2m}}\right)$$

We deduce that :

(a) For $k+j < 2m$ kernels are integrable for $x \in \partial\Omega$ and we can switch the integral and the limit symbols $\frac{\partial^k}{\partial n^k} V_j^-(x) = \frac{\partial^k}{\partial n^k} V_j(x)$.

(b) For $k+j > 2m$ thanks to [[8], Lemma 6.4], we get the same equality in the sense of the main Cauchy value.

By using (10.29), (10.31) is only possible if

$$\begin{cases} \dfrac{\partial^k}{\partial n^k} V_{2m-k}^-(F_{2m-k})(x) = \dfrac{(-1)^m}{2} F_{2m-k}(x) + \dfrac{\partial^k}{\partial n^k} V_{2m-k}(F_{2m-k})(x) \\ \dfrac{\partial^k}{\partial n^k} V_j^-(x) = \dfrac{\partial^k}{\partial n^k} V_j(x), \text{ for } 1 \leq j \leq 2m \text{ and } j \neq 2m-k \end{cases}$$

For $x \in B_R \backslash \overline{\Omega}$, (10.30) is changed in

$$w(x) = -\sum_{i=1}^{2m} V_i^+(F_i)(x) + G(x)$$

with $G(x)$ defined similarly by using multi-layer potentials on the boundary ∂B_R and regular in a neighbourhood of $\partial\Omega$. By the same reasoning we have

$$\begin{cases} \dfrac{\partial^k}{\partial n^k} V_{2m-k}^+(F_{2m-k})(x) = -\dfrac{(-1)^m}{2} F_{2m-k}(x) + \dfrac{\partial^k}{\partial n^k} V_{2m-k}(F_{2m-k})(x) \\ \dfrac{\partial^k}{\partial n^k} V_j^+(x) = \dfrac{\partial^k}{\partial n^k} V_j(x), \text{ for } 1 \leq j \leq 2m \text{ and } j \neq 2m-k \end{cases}$$

which proves the relations $(\mathcal{R}_1)$.

(ii) **Proof of relations** $(\mathcal{R}_2)$

Consider $0 \leq k \leq m-1$, similarly we get for $x \in \partial\Omega$:

$$\frac{A_k w(x)}{2} = (-1)^m \sum_{i=1}^{2m} A_k V_i(F_i)(x), \qquad x \in \partial\Omega$$

And by the same reasoning used for (i), and by using (10.29), this equality is only possible if

$$\begin{cases} A_k V^-_{m-k}(F_{m-k})(x) = -\dfrac{(-1)^m}{2} F_{m-k}(x) + A_k V_{m-k}(F_{m-k})(x) \\ \qquad A_k V^-_j(F_j)(x) = A_k V_j(F_j)(x), \text{ for } 1 \leq j \leq 2m \text{ and } j \neq m-k \end{cases}$$

By the same way, we get

$$\begin{cases} A_k V^+_{m-k}(F_{m-k})(x) = \dfrac{(-1)^m}{2} F_{m-k}(x) + A_k V_{m-k}(F_{m-k})(x) \\ \qquad A_k V^+_j(F_j)(x) = A_k V_j(F_j)(x), \text{ for } 1 \leq j \leq 2m \text{ and } j \neq m-k \end{cases}$$

which ends the proof of the relations $(\mathcal{R}_2)$ and the lemma's one.

□

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